Maria Trybuła

THE BERGMAN KERNEL FUNCTION AND RELATED TOPICS

PhD thesis

Advisor:
Professor Włodzimierz Zwonek

Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Mathematics
Kraków 2015

homo sum, humani nihil a me alienum puto
Publius Terentius Afer

Contents

Introduction

This work refers to the geometric function theory. It contains two slightly connected parts. Roughly speaking, the main role in the first one is played by the Bergman invariants, and in the second one by the Kobayashi distance.

Examples of kernels have been known for a long time. But the characteristic properties of these kernels as we now understand them have only been stressed and applied since the begining of the XXth century. There have been and continue to be two trends in the considerations of these kernels. One of them, which we are interested in, is to find the corresponding kernel K for a class of functions. It was initiated during the first decade of the twentieth century in the work of Zaremba on boundary value problems for harmonic and biharmonic functions. **Zaremba was the first to introduce, in a particular case, the kernel corresponding to a class of functions, and to state its reproducing property** (cf. [**Aro**] or [**Sza**]). However, he did not develop any general theory, nor did he give any particular name to the kernels he introduced. It appears that nothing was done in this direction until the third decade of XXth century (cf. [**Sza**]). Then a Polish-born American mathematician Stefan Bergman inspired by Schmidt's lectures in [**Ber1**] began the study of the space of square integrable holomorphic functions on a domain D in $\mathbb{C}^n$. He noticed that it has two useful features: completeness (in $L^2(D)$) and, what is more interesting, has the reproducing property. 20 years later Aronszajn proved that this is a characteristic property of every functional Hilbert space, i.e. we are able to recover each element of the space with the kernel functions' help (cf. [**Aro**]). In this way the interests of specialists in functional analysis and complex analysis have met.

The Bergman kernel is a canonical kernel that can be defined on any bounded domain. It has wonderful invariance properties, leading to the Bergman metric and the Bergman distance. The metric was the first-ever Kähler metric, and that in turn originated a new branch in complex differential geometry. Moreover, the Bergman kernel has certain extremal properties that make it a powerful tool in the theory of partial differential equations (cf. [**Ber3**]). Also the form of the singularity of the Bergman kernel (calculable for some interesting classes of domains) explains many phenomena of the function theory of several complex variables.

As it usually happens in such general constructions, the kernel is hard to calculate explicitly. One approach to calculate the kernel function is as follows. Find any orthonormal basis and then try to sum a series. In general, it is a long and not easy way. However, there are well known situations where this method is efficient, i.e. when we are able to find an explicit formula for the sum of the series. Of course, the unit ball, polydisc in $\mathbb{C}^n$ or the Cartan domains are classical ones (cf. e.g.[**Jar-Pfl2**] and [**Hua**]). Perhaps, a more complex example is contained in a recent paper [**M-R-Z**].

Using the techniques from the representation theory, Misra, Roy and Zhang computed the reproducing functions for some classes of weighted Bergman spaces on the symmetrized polydisc (in this way they rediscovered [**Edi-Zwo**, Proposition 9]).

The Bergman metric is the kernel's 'derivative'. Consequently, calculating it is more complicated and there are even less known examples of domains for which the metric is known in a clear form. Probably, the only ones are: the Euclidean ball, the minimal ball, the polydisc and some special ellipsoids (cf. [**Jar-Pfl2**], [**Pfl-You**]). In Section 1.1 we adopt some ideas from [**M-R-Z**] and add to the list $\mathbb{G}_2$, the object which has many applications ([**Try3**]).

The symmetrized polydisc appeared in the theory of μ-synthesis (cf. [**Agl-You1**], [**A-Y-Y**]) and turned out to be an important object in the geometric function theory (cf. [**Cos1**]). Its two dimensional counterpart $\mathbb{G}_2$ called the symmetrized bidisc because of its interesting properties was intensively investigated by many authors, among others Agler, Costara, Edigarian, Jarnicki, Nikolov, Pflug, Thomas, Young, Zwonek (some of the papers are listed in references). It seems to play an important role not only in complex analysis (it is the first known example of non-biholomorphic to a convex domain but hyperconvex for whose the Lempert Theorem holds) but also in solving Pick-Nevanlinna Interpolation Problem for $n = 2$ (cf. [**Agl-You2**]). Because of this, the symmetrized polydisc, will be mentioned very often throughout this work.

Misra, Roy, and Zhang [**M-R-Z**] recently studied the pullback of the Bergman space under a proper holomorphic mapping in the context of the symmetrized polydisc. In Section 1.2 we show the generalized construction for arbitrary domains. In particular, we obtain a new proof of Bell's transformation formula for the Bergman kernel function under proper holomorphic mappings (Corollary 1.2.4). Comparing with the original proof, our proof is closer to functional analysis. However, at the climax we use the same tool, i.e. the Riemann Riemovable Singularity Theorem.
As an application in Section 1.3 we demonstrate that the Bergman kernel function of the tetrablock has zeros. To our surprise it turned out that the tetrablock is not a Lu Qi Keng domain. Moreover, it vanishes at very simple points. The tetrablock was first studied in [**A-W-Y**]. Afterwards it was studied by many authors (cf. e.g. [**E-K-Z**], [**Zwo**]). In particular, it was shown that the tetrablock is a $\mathbb{C}$-convex domain (see [**Zwo**]). The importance of the tetrablock for geometric function theory follows from the fact that it is the second example (the first one was the symmetrized bidisc) which is hyperconvex and not biholomorphically equivalent to a convex domain but despite it the Lempert Theorem holds for it (see [**E-K-Z**]).

In Section 1.4 we show an extended version of [**Rud1**, Proposition 2.1]. We show that every holomorphic mapping F is a proper map onto its image if it admits a finite group $\mathcal{U}$ of automorphisms under which F is precisely $\mathcal{U}$-invariant. In such a general setting it could be applied among others to the above mentioned case of the tetrablock and the symmetrized polydisc. This will help us to avoid the ad hoc proofs of properness and openness of a wide class of mappings (see remarks following Proposition 1.4.1).

In Section 1.5 we study the behavior of the Bergman distance on Dini smooth domains. We complete the work started by Nikolov in [**Nik1**] and show that the natural estimates on Dini-smooth domains for the Carathéodory or the Kobayashi distances remain true for the Bergman distance. Poposition 1.5.6 is a non-trivial generalization of [**Nik1**, Proposition 8]. Non-trivial because the given proof in Section 1.5 uses a very deep result by Balogh-Bonk (see [**Bal-Bon**]) about boundary behavior of the Kobayashi distance. Moreover, the proof applies the localization of the Kobayashi metric obtained by Forstneric-Rosay in [**For-Ros**].

The last problem which we consider in Section 1.5 is also devoted to the comparison of invariant distances. The classical results by Graham and Venturini say that the Kobayashi distance is comparable with the Carathéodory distance on sctrictly pseudoconvex domains near the boundary (cf. e.g. [**Jar-Pfl2**, Chapter 10] and references there). We show that for a fixed point z in a finitely connected planar domain D

$$b_D(z,\cdot) \sim \sqrt{2}c_D(z,\cdot) \sim \sqrt{2}k_D(z,\cdot)$$

near the boundary ∂D (Proposition 1.5.17), i.e. near the boundary all distances behave like on the unit disc.

Chapter 1 ends with the characterization of $\mathbb{G}_n$. Proposition 1.6.10 is a generalization of some of Costara's results from [**Cos2**]. The proofs utilize notion of the polar derivative.

In the first part of Chapter 2 we study balls with respect to the Kobayashi distance and metaphorically speaking we try to describe how big or small they are (see the remark following Theorem 2.1.3). Equivalently, we approximate from inside and outside the balls by analytic polydiscs. The radiuses we express in terms of the natural parameters given by the minimal basis. Our the only toolkit is the so-called minimal-basis which one might define for any domain in $\mathbb{C}^n$. The first place where the construction considered is the paper by McNeal [**McN**]. However, we apply its modified version given in [**Nik-Pfl**]. The proof of Theorem 2.1.3 is purely geometric and uses substantially the geometric properties of a domain.

The first results in a spirit of Theorem 2.1.3 can be found in [**Ala**, Theorem 1 and Theorem 5.1], where the strongly pseudoconvex case in $\mathbb{C}^n$ and the weakly pseudoconvex finite type case in $\mathbb{C}^2$ are discussed with applications to invariant forms of Fatou type theorems (for the boundary values). The weakly pseudoconvex finite type case in $\mathbb{C}^2$, as well as the convex finite type case in $\mathbb{C}^n$, are treated in [**Mah**, Propositions 8.8 and 8.9] as byproducts of long considerations. The strongly pseudoconvex case in $\mathbb{C}^n$ and the weakly pseudoconvex finite type in $\mathbb{C}^2$ are particular cases of the pseudoconvex Levi corank one case which are considered in respectively [**Her**] and [**B-M-V**]. The behavior of the Kobayashi balls in all the mentioned results is given in terms of the Levi geometry of the boundary which is assumed smooth and bounded. Comparing with all mentioned results the proof given here is short and purely geometric. Besides, Section 2.1 contains the local version of Theorem 2.1.3.

In Section 2.2. we discuss a non-classical notion of hyperbolicity which is rare to meet in the classical literature on the holomorphically invariant distances (by the classical hyperbolicity we understand the notion given in [**Jar-Pfl2**, Chapter 2.3, Chapter 3.3]). We are just discussing Gromov hyperbolicity (we refer the reader to [**Gro1**] or [**Väi**] for an elegant general account on the theory). The prototype of a Gromov hyperbolic space is a simply connected complete Riemannian manifold with curvature bounded above by a negative constant (cf. [**Gro1**, pg. 76]). In the model situation the metric is assumed to be at least $\mathcal{C}^2$. Moreover, it is often assumed that the space on which we are working is geodesic, i.e. we are able to join every two points by a curve which realizes the distance between them. However, such a strict approach is not necessary. Actually, we might replace geodesics by "*almost geodesics*" and consider intrinsic spaces instead of geodesic spaces. Recall that a curve $\alpha : [0,1] \to D$ is called an ϵ-almost geodesic if its lenght does not exceed $\epsilon + d(\alpha(0), \alpha(1))$, where (D, d) is a metric space and $\epsilon > 0$. If for every two points in (X, d) and every choice of a positive ϵ there exists ϵ-almost geodesic than we say that (X, d) is the intrinsic metric space. However, the result of Bonk and Schramm [**Bon-Sch**, Theorem 4.1] stating that every hyperbolic metric space can be isometrically embedded into a hyperbolic geodesic space implies that in fact we did not change the class of spaces. And here by [**Jar-Pfl2**, Proposition 3.3.1] we meet the Kobayashi distance(!).

In some sense we even measure the Gromov hyperbolicity, and express it by saying 'X is $\delta-$hyperbolic'. The number δ gives some information about the geometry of the space. If the curvature K of the model space satisfies

$$K \leq -\delta^2 < 0,$$

then the space is $(C\delta)-$hyperbolic, where $C \in (1, 10)$ (cf. [**Gro1**, 1.5(1)]). In other words, a space (X, d) is δ-hyperbolic if every geodesic triangle is δ-thin (for the precise formulation see Section 2.2).

There is no connection between the ordinary hyperbolicity and the Gromov one (in both cases with respect to the Kobayashi distance). Some 'perfect' domains in $\mathbb{C}^n$, for instance bounded and convex, are hyperbolic with respect to the Kobayashi distance, but not hyperbolic in the Gromov's sense (cf. Theorem 2.2.13, Proposition 2.2.11).

The first work concerning Gromov hyperbolicity on domains endowed with the Kobayashi distance was given by Balogh and Bonk [**Bal-Bon**] who gave both positive and negative examples. Among other results, they proved that the Cartesian product of strictly pseudoconvex domains is not Gromov hyperbolic. As an immediate consequence we obtain that polydisc is not hyperbolic. Moreover, even its "symmetrized" counterpart is not ([**N-T-T**]). Buckley in [**Buc**], following Bonk, claimed that it is because of the flatness of the boundary rather than the lack of smoothness that Gromov hyperbolicity fails. Recently, Gaussier and Seshadri have provided a proof of that conjecture. More precisely, their main result in [**Gau-Ses**] states that any bounded convex domain in $\mathbb{C}^n$ whose boundary is $\mathcal{C}^\infty$-smooth and contains an analytic disc, is not Gromov hyperbolic with respect to the Kobayashi distance. In [**Gau-Ses**] the $\mathcal{C}^\infty$ smoothness assumption is essential. Our aim is to prove this result in a shorter way in $\mathbb{C}^2$, assuming only $\mathcal{C}^{1,1}$-smoothness. Moreover, the proofs of the facts we use are more elementary. Besides, we give a partial answer to the question raised in [**Bal-Bon**], see Theorem 2.2.13. Namely, we show under certain assumption that a bounded convex domain in $\mathbb{C}^2$ which satisfies an 'infinite' type condition is not Gromov hyperbolic.

The question whether there is any connection between Gromov hyperbolicity and pseudoconvexity naturally arises. The known examples do not say anything in this matter. Also, it is easy to construct domains which are Gromov hyperbolic but neither pseudoconvex nor smooth. Proposition 2.2.19 yields, in particular, a family of non pseudoconvex domains with smooth boundaries which are Gromov hyperbolic.

We finish Chapter 2 with a remark on the Kobayashi metric of $\mathbb{G}_2$.

Finally, in the appendix we have included the most important facts relating to the objects under consideration which we assume as known to the Reader. At the end, we present, for the convenience of the Reader, the list of symbols used throughout the work and the list of works cited.

The accent falls onto authors' own results taken from the papers [**Try1**], [**Try2**], [**Try3**] and from the joint papers with N. Nikolov [**Nik-Try1**], [**Nik-Try2**], and with N. Nikolov, P. Thomas [**N-T-T**].

Acknowledgments

I would like to thank my advisor Professor Włodzimierz Zwonek for his guidance, instant support and inspiring discussions during last years.

I am also very grateful to Professor Nikolai Nikolov for introducing me to the world of the Kobayashi distance. Many results contained in this work are the fruits of our joint trip into this rich and beautiful world.

During my studies I had the opportunity to visit the University of Toulouse (France), the Carl von Ossietzky University (Germany), as well as the Bulgarian Academy of Science (Bulgaria). These visits enriched my mathematical (and not only mathematical) view point and contributed a lot to the creation of this thesis. Thus I wish to thank Professor Pascal Thomas, Professor Peter Pflug, Professor Nikolai Nikolov and the Institutions for the hospitality during my stays.

Having the great possibility to be a follower of Professor Marek Jarnicki, Professor Zbigniew Błocki, Professor Jan Stochel, Doctor Piotr Niemiec and Doctor Zdzisław Pogoda I have learned a lot from them. To them, as well as to all my colleagues from the Institute of Mathematics of the Jagiellonian University I send words of sincere gratitude.

The author was supported by the National Science Center grant **PRO-2013/11/N/ST1/03609** and by the Foundation for Polish Science grant **MPD/2010/1**.

CHAPTER 1

On the Bergman theory

The original books by Bergman ([**Ber2**], [**Ber3**]) are still a relevant and a good source on the Bergman theory. Among the new ones, it is worth to mention Jarnicki-Pflug's monograph [**Jar-Pfl2**, Chapter 6]. Perhaps, expositions by Krantz [**Kra1, Kra2**] give rise to controversy but they are very readable for the beginers and this is the only reason why their names appear here.

1.1. The Bergman metric on $\mathbb{G}_2$

Before we formulate the main theorem of this Section, we introduce a few definitions.

Fix $n \in \mathbb{N}$.

Definition 1.1.1. For $l \in \mathbb{N}$ let s_l be the *elementary symmetric function* of degree l. In other words s_l is the sum of all products of l distinct variables z_k

$$s_l(z) := s_l(z_1, \dots, z_n) := \sum\nolimits_{1 \le k_1 < \dots < k_l \le n} z_{k_1} \cdots z_{k_l}.$$

Let $s : \mathbb{C}^n \to \mathbb{C}^n$ be the function of symmetrization given by the formula

$$s(z_1, \dots, z_n) := \big(s_1(z_1, \dots, z_n), \dots, s_n(z_1, \dots, z_n)\big).$$

Recall that the map $s|_{\mathbb{D}^n} : \mathbb{D}^n \to s(\mathbb{D}^n) =: \mathbb{G}_n$ is a proper holomorphic map (see Proposition 1.4.1 and the remarks following it) and its image $\mathbb{G}_n$ is called the *symmetrized polydisc*.

Fix a domain $G \subset \mathbb{C}^n$.

Definition 1.1.2. We put

$$\mathbb{A}^2_\alpha(G) := \left\{ f \in \mathcal{O}(G) : \int_G |f|^2 \alpha \, dV < \infty \right\},$$

and

$$L^2_\alpha(G) := \left\{ f : G \to \mathbb{C} \; : f \text{ is Lebegue measurable}, \; \int_G |f|^2 \alpha \, dV < \infty \right\},$$

where dV stands for $2n$ dimensional Lebegue measure, and $\alpha : G \to (0, \infty)$ is a positive continous function. The space $\mathbb{A}^2_\alpha(G)$ with the scalar product

$$\langle f, g \rangle_{\mathbb{A}^2_\alpha(G)} = \int_G f(z) \overline{g(z)} \alpha(z) \, dV(z), \quad f, g \in \mathbb{A}^2_\alpha(G)$$

is a complex Hilbert space, the Hilbert space of all square integrable holomorphic functions on G with respect to the weight α. The functional analysis specialists called that kind of spaces as $\mathbb{A}^2_\alpha(G)$ *functional Hilbert spaces*. This name comes from the fact that the elements of the space are functions, not equivalence classes. Its deepest consequence is the fact that the convergence in $\mathbb{A}^2_\alpha(G)$ understood as a Hilbert space implies the local uniform convergence on G. The idea of Theorem 1.1.4 arose out of this simple observation.

The *Bergman kernel with weight* α on D is given by the formula

$$K^\alpha_G(z,w) := \sum_{j\in J} \varphi_j(z)\overline{\varphi_j(w)}, \quad z,w \in G, \tag{1.1.1}$$

where $\{\varphi_j\}$ is an orthonormal basis for $\mathbb{A}^2_\alpha(G)$, J is at most countable set.(1) For $\alpha \equiv 1$ we simply write $K^\alpha_G = K_G$, and call it *the Bergman kernel function.* For a fixed $w \in G$ a function $K^\alpha_G(\ ,w)$ is an element of $\mathbb{A}^2_\alpha(G)$.

We make one very important remark on the right side of (1.1.1). The series sums locally uniformly on $G\times G$ to the Bergman kernel K^α_G (for details cf. [**Kra2**, Proposition 1.4.7]). In particular, by the Hartogs Theorem

$$K^\alpha_G \in \mathcal{C}^\omega(G\times G). \tag{1.1.2}$$

From now until the end of the current Section we set $\alpha \equiv 1$ and $G \subset \mathbb{C}^n$ is assumed to be a bounded domain.

Definition 1.1.3. The *Bergman metric* is given by the equality

$$\beta^2_G(z;X) := \sum_{1\le j,k\le n} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log K_G(z,z) X_j \overline{X}_k, \qquad z,\, w\in G,\ X\in\mathbb{C}^n. \tag{1.1.3}$$

We postpone for a while the proof of the fact that β_G is a metric (see also [**Jar-Pfl2**, Chapter 6]). Moreover, $\beta^2_G \in \mathcal{C}^\omega(G\times\mathbb{C}^n)$.

This definition of metric is not a good tool for explicit calculations. Therefore, we use another approach, which is equivalent to the above one. The alternative description is as follows

$$\beta_G(z;X) := \frac{1}{\sqrt{K_G(z,z)}} \sup\{|f'(z)X| : f \in \mathcal{O}(G), f(z)=0, \|f\|_{\mathbb{A}^2(G)} \le 1\}, \qquad z\in G,\ X\in\mathbb{C}^n. \tag{1.1.4}$$

For the equivalence of Definition 1.1.2 and Definition 1.1.3 cf. [**Jar-Pfl2**, Theorem 6.2.5].

The boundedness of G implies that a function identically equal to 1 and the coordinates functions are elements of $\mathbb{A}^2(D)$. Consequently, the right side of (1.1.4) actually defines a positive number.

Put

$$M_G(z;X) := \sup\{|f'(z)X| : f \in \mathcal{O}(G), f(z)=0, \|f\|_{\mathbb{A}^2(G)} \le 1\}, z\in G,\ X\in\mathbb{C}^n. \tag{1.1.5}$$

(1)Since $L^2_\alpha(G)$ is a separable Hilbert space, then so is its subspace $\mathbb{A}^2_\alpha(G)$. Thus there is at most countable, complete orthonormal basis $\{\varphi_j\}$ for $\mathbb{A}^2_\alpha(G)$.

One of the most important properties of the Bergman metric is that it is invariant under biholomorphic mappings, i.e.

$$\beta_D(z;X) = \beta_G(F(z);F'(z)X), \tag{1.1.6}$$

where $F : D \to G$ biholomorphic map, $z \in D \subset \mathbb{C}^n$, $X \in \mathbb{C}^n$. For a coresponding property for the Bergman kernel function see the next Section. In fact, the property (1.1.6) is an instantenous consequence of (1.1.4) and Corollary 1.2.4.

Now we are ready to state our first result

Theorem 1.1.4. ([**Try3**]) *For $s_2 \,\epsilon\, [0,1)$, $X = (X_1, X_2) \in \mathbb{C}^2$ we have the following equality*

$$\beta_{\mathbb{G}_2}\Big((0,s_2);(X_1,X_2)\Big) = \sqrt{B_1|X_1|^2 + B_2|X_2|^2}\,,$$

where $B_1 = \frac{4x^2+10x+2}{(1-x)^2(3x+1)}$, $B_2 = \frac{39x^2+18x+7}{(1-x)^2(3x+1)^2}$, $x = s_2^2$.

Before we proceed to the proof of Theorem 1.1.4 we need to make several observations.

For a non empty set A (with no ordering), we put

$$l^2(A) := \{(x_a)_{a\epsilon A} : x_a \,\epsilon\, \mathbb{C},\ \sum_{a\epsilon A} |x_a|^2 < \infty\}.$$

One can naturally embed $l^2(B)$ in $l^2(A)$ if $B \subseteq A$. Observe that $l^2(A)$ is the Hilbert space of all square summable sequences in $\mathbb{C}^A$ with the standard scalar product

$$\langle x, y\rangle := \sum_{a\epsilon A} x_a \overline{y_a}$$

(the reasoning is the same as in $l^2 = \{(x_j)_{j=1}^{\infty} : \sum |x_j|^2 < \infty\}$ case). Clearly, the canonical embeding $l^2(B) \subset l^2(A)$ is an isometry.

In the proof of Theorem 1.1.4 Schur polynomials, which up to some constants form an orthonormal basis for $\mathbb{A}^2(\mathbb{G}_n)$, appear (see [**M-R-Z**]). Schur polynomials are defined in terms of partitions. We call a finite sequence $p = (p_1, \ldots, p_n)$ of decreasing (not necessarily strictly) non-negative integers *a partition* (n is called the length of p). By $[n]$ denote the set of all the partitions of length n. Let $\delta := (n-1, \ldots, 0)$, $[[n]] := [n] + \delta$.

We shall need constants

$$c_p^2 = \frac{p_1(p_2+1)}{\pi^2},$$

and Schur polynomials are

$$S_p(z) = \frac{a_p(z)}{a_\delta(z)},\ z \,\epsilon\, \mathbb{D}^2,$$

where $a_p(z) := z_1^{p_1} z_2^{p_2} - z_1^{p_2} z_2^{p_1}$, $p \,\epsilon\, [[2]]$ (it is a very special case of a more general situation - cf. [**Ful-Har**]). Elementary calculation shows that S_p is actually a polynomial. Additionally, define $S_p = 0$, if $p \in \mathbb{Z}^2 \setminus [[2]]$. From [**M-R-Z**] we know that the set

$$\{e_p = c_p S_p : p \,\epsilon\, [[2]]\}$$

is the complete orthonormal system for $\mathbb{A}^2(\mathbb{G}_2)$.

There are some relations between Schur polynomials and elementary symmetric functions, called *the Jacobi-Trudy identities* (cf. [**Ful-Har**]). To understand them, we introduce the notion of a conjugate partition. If $\lambda = (\lambda_1, \dots, \lambda_n) \epsilon [n]$, then *a conjugate partition* to λ is a partition $\mu = (\mu_1, \dots, \mu_{\lambda_1})$ (denoted by λ') such that $\mu_k = \#\{j : \lambda_j \geq k\}$. Observe that the length of μ depends on λ_1.

The Jacobi-Trudy identities are described as following

$$S_{p+(1,0)} = \det \begin{pmatrix} \mathbb{I}_{p_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}, \tag{1.1.7}$$

where $\mathbf{A}$ is $(p_1 - p_2)$ square matrix,

$$\mathbf{A} = \begin{pmatrix} s_1 & s_2 & 0 & \dots & 0 \\ 1 & s_1 & s_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \dots & s_2 \\ 0 & 0 & \vdots & 1 & s_1 \end{pmatrix}$$

(cf. [**Ful-Har**, Appendix A] for the general version of the identities).

A straightforward conesquence of (1.1.6) is contained in the Lemma 1.1.5.

Lemma 1.1.5.

$$S_{(k+m,m)+\delta}(s_1, s_2) = s_2^m \sum\nolimits_{l=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^l \binom{k-l}{l} s_1^{k-2l} s_2^l, \text{ for } k,\, m \geqslant 0,$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Proof of Theorem 1.1.4. **Step I.** Recall that the group of automorphisms of the the symmetrized bidisc consists of the mappings of the form

$$H(z_1 + z_2, z_1 z_2) = (s_1(h(z_1), h(z_2)), s_2(h(z_1), h(z_2))), \quad z_1, z_2 \,\epsilon\, \mathbb{D}, \tag{1.1.8}$$

where $h \,\epsilon\, \mathrm{Aut}(\mathbb{D})$ (cf. [**Jar-Pfl1**]). Therefore, by (1.1.6) it is enough to compute $\beta_{\mathbb{G}_2}$ at points $(0, s_2)$, $s_2 \in [0, 1)$ (with a little abuse of notation till the end of the Section 1.1 s_2 denotes a number in $[0, 1)$; we hope it will not cause any problem for the Reader). Actually, for our purposes it is enough to observe that maps of the form (1.1.8) are automorphisms of $\mathbb{G}_2$.

Step II. Remind that

$$\beta_{\mathbb{G}_2}\Big((0, s_2); X\Big) = \frac{1}{\sqrt{K_{\mathbb{G}_2}((0, s_2), (0, s_2))}} \times \\ \sup\{|f'(0, s_2)(X)| : f \,\epsilon\, \mathbb{A}^2(\mathbb{G}_2),\ f(0, s_2) = 0,\ \|f\|_{\mathbb{A}^2(\mathbb{G}_2)} \leqslant 1\},$$

for $s_2 \in [0, 1)$, $X \in \mathbb{C}^2$. Let:

$$x_0 = \left\{ \left((-1)^n \sqrt{\frac{(k + 2n + 1)(k + 1)}{\pi^2}}\ s_2^{n+k} \right)_{(k+2n+1,k)} \right\}_{k,\, n \geqslant 0},$$

and

$$c = \left\{ \left((-1)^n (n+k) \sqrt{\frac{(k+2n+1)(k+1)}{\pi^2}} \; s_2^{n+k-1} \overline{X_2} \right)_{(k+2n+1,k)} \right\}_{k,\, n \geqslant 0}$$
$$\cup \left\{ \left((-1)^n (n+1) \sqrt{\frac{(k+2n+2)(k+1)}{\pi^2}} \; s_2^{n+k} \overline{X_1} \right)_{(k+2n+2,k)} \right\}_{k,\, n \geqslant 0}.$$

c induces a bounded operator

$$\Lambda : l^2([[2]]) \to \mathbb{C},$$
$$\Lambda(z) = \langle z, c \rangle .$$

Fix $f = \sum_{p \,\epsilon\, [[2]]} \alpha_p c_p S_p$ from $\mathbb{A}^2(\mathbb{G}_2)$. Let $\alpha = \{\alpha_p\}_{p \in [[2]]}$. Observe that

$$f'(0, s_2)X = \Lambda(\alpha).$$

Consequently, the supremum which appears in the formula of $\beta_{\mathbb{G}_2}$ is equal to

$$\sqrt{K_{\mathbb{G}_2}((0,s_2),(0,s_2)}\; \beta_{\mathbb{G}_2}((0,s_2);X) = \|\Lambda|_{\{x_0\}^\perp}\|$$

(the operator norm of Λ restricted to $\{x_0\}^\perp$). Before we write the explicit formula for that supremum, we state some consequence of the Riesz Representation Theorem and the Pythagorean Theorem.

Lemma 1.1.6. *Let $\Lambda : H \to \mathbb{C}$ be a bounded linear functional on a Hilbert space H. Assume $H = \bigoplus_{1 \le j \le n} H_j$, $n \in \mathbb{Z}_{>0}$. That is, H is a direct sum of pairwise orthogonal subspace H_j of H. Then,*

$$\|\Lambda\|^2 = \sum_{1 \le j \le n} \|\Lambda|_{H_j}\|^2.$$

Moreover, the statement remains true when $H = \bigoplus_{j=1}^{\infty} H_j$.

So,

$$\|\Lambda|_{\{x_0\}^\perp}\|^2 = \|\Lambda\|^2 - \|\Lambda|_{\mathrm{span}\{x_0\}}\|^2 = \langle c, c\rangle - \frac{\left|\langle c, x_0\rangle\right|^2}{\langle x_0, x_0\rangle}.$$

Put $x = s_2^2$. To finish the proof, it is enough to find the remaining scalar products:

$$\langle x_0, x_0\rangle = K_{\mathbb{G}_2}((0,s_2),(0,s_2)) = \frac{3x+1}{\pi^2(1-x)^4},$$

$$\langle c, c\rangle = \frac{4x^2+10x+2}{\pi^2(1-x)^6}\,|X_1|^2 + \frac{27x^2+46x+7}{\pi^2(1-x)^6}\,|X_2|^2,$$

$$|\langle c, x_0|^2\rangle = \frac{(9x+7)^2 x}{\pi^4(1-x)^{10}}\,|X_2|^2.$$

□

The methods presented above also work on any domain in $\mathbb{C}^n$ but possibly not in so efficient way as for the symmetrized bidisc. Finding the concise formula causes difficulties, even for the higher dimensional counterparts of $\mathbb{G}_2$.

1.2. Projection and Bell's formulas

In the introduction we mentioned that the Bergman kernel function has the reproducing property. Until now the reproducing character of the Bergman kernel has not been displayed yet. Formula (1.1.1) apparently hides it. In the present Section we will make use of it.

Let us start with a formula for the Bergman kernel, equivalent to (1.1.1).

Let $G \subset \mathbb{C}^n$ be a domain and let $\alpha : G \to \mathbb{R}_{>0}$ be a strictly positive continuous function.

Cauchy integral formula implies that for every $z \in G$ the evaluation functional

$$ev_z : \mathbb{A}^2_\alpha(G) \ni f \to f(z) \in \mathbb{C}$$

is continuous. Thus from the Riesz Representation Theorem there is the unique function $K^\alpha_{G,\,z} \in \mathbb{A}^2_\alpha(G)$ (called *the kernel function*) such that

$$ev_z(f) = \langle f, K^\alpha_{G,\,z}\rangle_{\mathbb{A}^2_\alpha(G)}.$$

Then *the Bergman kernel function with weight* α might be written as follows

$$K^\alpha_G(z,w) := \langle K^\alpha_{G,\,w}, K^\alpha_{G,\,z}\rangle_{\mathbb{A}^2_\alpha(G)}, \quad z,\, w \in G. \tag{1.2.1}$$

Let $\pi : D \to G$ be a proper holomorphic map with multiplicity m (D, G are domains in $\mathbb{C}^n$).

Motivated by [**Lig**] and [**M-R-Z**], we investigate the relations between weighted Bergman Spaces: $\mathbb{A}^2_\alpha(G)$ and $\mathbb{A}^2_{\alpha\circ\pi}(D)$.

We proceed to formulate the most important component of the next theorem.

Observe that π induces an operator Γ, where

$$\Gamma : \mathbb{A}^2_\alpha(G) \to \mathbb{A}^2_{\alpha\circ\pi}(D),$$

is defined as follows

$$\Gamma f = \frac{1}{\sqrt{m}}(f\circ\pi)J\pi, \quad f \in \mathbb{A}^2_\alpha(G).$$

Γ's adjoint operator is of great importance to us, so we explain how it works. In fact, Γ^* equals its inverse Γ^{-1} (if Γ is understood as an operator from $\mathbb{A}^2_\alpha(G)$ onto $\Gamma\mathbb{A}^2_\alpha(G)$). To describe Γ^* take any $g \in \Gamma\mathbb{A}^2_\alpha(G)$. Then $\frac{g}{J\pi}$ is a well-defined function on a dense, open subset of D (the set of regular points of π). Moreover, notice that $\frac{g}{J\pi}$ is invariant under π, that is $\frac{g}{J\pi}(z) = \frac{g}{J\pi}(w)$ for any $z,\, w \in D$ such that $\pi(z) = \pi(w)$, $J\pi(w), J\pi(z) \neq 0$. Therefore, equality $\widetilde{(\frac{g}{J\pi})}(\pi(z)) = \frac{g}{J\pi}(z)$ defines well a holomorphic function on G except for the (analytic) set of critical values of π. However, the Riemann Removable Singularity Theorem for square integrable holomorphic functions (cf. [**Jar-Pfl2**, Theorem 4.2.9]) ensures that $\widetilde{(\frac{g}{J\pi})}$ has a holomorphic extension on D (denoted by the same symbol). After this consideration the adjoint operator to Γ might be described by the equality

$$\Gamma^* g = \sqrt{m}\widetilde{(\frac{g}{J\pi})}, \quad g \in \Gamma\mathbb{A}^2_\alpha(G).$$

Theorem 1.2.1. ([**Try1**]) *The set* $\Gamma\mathbb{A}^2_\alpha(G)$ *is a closed subspace of* $\mathbb{A}^2_{\alpha\circ\pi}(D)$ *that is isometrically isomorphic with* $\mathbb{A}^2_\alpha(G)$ *via* Γ. *The orthogonal projection* P *onto*

$\Gamma\mathbb{A}^2_\alpha(G)$ *is given by a formula*

$$Pg = \frac{1}{m}\sum_{k=1}^{m}(g\circ\pi^k\circ\pi)J(\pi^k\circ\pi),\quad g\in\mathbb{A}^2_{\alpha\circ\pi}(D),$$

where $\{\pi^j\}_{j=1}^m$ *are the local inverses to* π*.*

Note that it will follow from the proof that the formula on the right side actually defines a function from $\Gamma\mathbb{A}^2_\alpha(G)\subset\mathbb{A}^2_{\alpha\circ\phi}(D)$.

Remark 1.2.2. In [**Jar-Pfl2**] the Riemann Removable Singularity Theorem is proved for $\alpha\equiv 1$, but this proof might be repeated without any trouble in case $\alpha\in\mathcal{C}(G,\mathbb{R}_{>0})$. In fact, the local boundedness of α (from below and above) is sufficient.

Remark 1.2.3. Let $\alpha\equiv 1$. The proof of Theorem 1.2.1 shows that Γ is the restriction of the operator Γ_e from $L^2(G)$ to $L^2(D)$, given by the same formula as Γ, and all statements contained in Theorem 1.2.1 hold for Γ_e. Moreover, this together with the transformation formula for the Bergman projection operator, given in [**Bell**], allows us to write $P_D\Gamma_e=\Gamma_eP_G$, where P_G and P_D denote Bergman projections of the domains D and G.

Proof of Theorem 1.2.1. The idea of the formula of Γ was inspired by the transformation rule

$$m\int_G f\alpha\,dV=\int_D (f\circ\pi)|J\pi|^2(\alpha\circ\pi)\,dV$$
$$\text{for any } f\in L^1_\alpha(G)=\left\{g:G\to\mathbb{C}\;:\;\int_G|f|\alpha\,dV<\infty\right\},$$

which makes the Γ an isometry. The above rule ensures that the range of Γ is a closed Hilbert subspace of $\mathbb{A}^2_{\alpha\circ\pi}(D)$. Therefore, Γ is a unitary operator from $\mathbb{A}^2_\alpha(G)$ onto $\Gamma\mathbb{A}^2_\alpha(G)$.

Thus, there is the orthogonal projection P from $\mathbb{A}^2_{\alpha\circ\pi}(D)$ onto $\Gamma\mathbb{A}^2_\alpha(G)$. We prove that P is given by the formula

$$Pg = \frac{1}{m}\sum_{k=1}^{m}(g\circ\pi^k\circ\pi)J(\pi^k\circ\pi),\quad g\in\mathbb{A}^2_{\alpha\circ\pi}(D).$$

Let us denote the right side by Qg. First of all we need to show that Q is well defined. Using the properness of π and the Schwarz inequality one can easily compute

$$\|Qg\|^2_{\mathbb{A}^2_{\alpha\circ\pi}(D)}=\frac{1}{m^2}\int_D\Big|\sum_{k=1}^{m}(g\circ\pi^k\circ\pi)J(\pi^k\circ\pi)\Big|^2(\alpha\circ\pi)dV\le$$
$$\frac{1}{m}\int_D\sum_{k=1}^{m}|(g\circ\pi^k\circ\pi)J(\pi^k\circ\pi)|^2(\alpha\circ\pi)dV=\|g\|^2_{\mathbb{A}^2_{\alpha\circ\pi}(D)},$$

for $g\in\mathbb{A}^2_{\alpha\circ\pi}(D)$. It remains to verify whether Qg is holomorphic. For that fix some $g\in\mathbb{A}^2_{\alpha\circ\pi}(D)$ and put $N(Jf):=(Jf)^{-1}(0)$, i.e. the zeros set of the Jacobian of f. Notice that the map $\frac{Qg}{J\pi}$ is a well defined holomorphic function on a set $D\setminus\pi^{-1}(\pi(N(J\pi)))$, constant on the fibres of π. So, it induces some map $\widetilde{(\frac{Qg}{J\pi})}$ which is holomorphic on $G \setminus \pi(N(J\pi))$. The Riemann Removable Singularity Theorem (see Remark 1.2.2) finishes the correctness of the definition of Q provided we know that $\widetilde{(\frac{Qg}{J\pi})}$ is square integrable with weight α on G. But for that it is enough to show that $Qg \in \mathbb{A}^2_{\alpha\circ\pi}(D)$ what we have just proved. Actually, we have established something more. Namely, that for any $g \in \mathbb{A}^2_{\alpha\circ\pi}(D)$ the equation $Qg = \Gamma f$ has solution f in $\mathbb{A}^2_{\alpha}(G)$. (The application of the Riemann Removable Singularity Theorem on the domain D would give us only that $Qg \in \mathbb{A}^2_{\alpha\circ\pi}(D)$.)

Secondly, notice that $Q^2 = Q$. Indeed,

$$
\begin{aligned}
Q^2 g = \frac{1}{m}\sum_{l=1}^{m}(Qg\circ\pi^l\circ\pi)J(\pi^l\circ\pi)
&= \frac{1}{m^2}\sum_{l=1}^{m}\sum_{k=1}^{m}(g\circ\pi^l\circ\pi\circ\pi^k\circ\pi)[J(\pi^l\circ\pi)\circ\pi^k\circ\pi]J(\pi^k\circ\pi)\\
&= \frac{1}{m^2}\sum_{k=1}^{m}\sum_{l=1}^{m}(g\circ\pi^l\circ\pi\circ\pi^k\circ\pi)[J(\pi^l\circ\pi)\circ\pi^k\circ\pi]J(\pi^k\circ\pi)\\
&= \frac{1}{m^2}\sum_{k=1}^{m}\sum_{l=1}^{m}(g\circ\pi^l\circ\pi)J(\pi^l\circ\pi\circ\pi^k\circ\pi)\\
&= \frac{1}{m^2}\sum_{k=1}^{m}\sum_{l=1}^{m}(g\circ\pi^l\circ\pi)J(\pi^l\circ\pi) = Qg,
\end{aligned}
$$

for $g \in \mathbb{A}^2_{\alpha\circ\pi}(D)$.

Up to this point, we only know that Q is the projection. Next, we proceed to show the equality $\operatorname{ran}\Gamma = \operatorname{ran} Q$. Similarly as above we get $Q\circ\Gamma = \Gamma$, which gives "$\subset$". It remains to demonstrate the opposite inclusion. So, the question is whether Q takes values in $\Gamma\mathbb{A}^2_{\alpha}(G)$. Since $Q^2 = Q$, it is enough to show that for any $g \in \mathbb{A}^2_{\alpha\circ\pi}(D)$ the equation $Qg = \Gamma f$ has solution f in $\mathbb{A}^2_{\alpha}(G)$, and it holds as we proved it before. Finally, since $Q^2 = Q$, $Q|_{\operatorname{ran} Q} = \operatorname{id}|_{\operatorname{ran} Q}$ and Q is bounded, Q is the orthogonal projection onto $\Gamma\mathbb{A}^2_{\alpha\circ\pi}(D)$. □

As a corollary of Theorem 1.2.1 we get Bell's Theorem (see [**Bell**]). Originally Bell formulated transformation rule for the Bergman kernel function with weight $\alpha \equiv 1$ for bounded domains. Here we shall prove that the same formula holds in a more general setting, which seems not to have been noticed in the literature. Moreover, our proof uses more functional analysis tools than Bell's proof, i.e. the fact that the Bergman kernel function is a kind of a reproducing kernel, not connecting the kernel so much with a space of functions for which it has a reproducing property.

Corollary 1.2.4. ([**Bell**], [**Try1**]) *Let D and G be domains in $\mathbb{C}^n$ and let $\pi : D \to G$ be a proper holomorphic map with multiplicity m. Denote by $\pi^1, \ldots, \pi^m$ the local* *inverses of* π. *Then*

$$\overline{J\pi(w)}K_G^\alpha(\pi(z),\pi(w)) = \sum_{k=1}^m K_D^{\alpha\circ\pi}(\pi^k\circ\pi(z),w)J\pi^k(\pi(z)),$$

$$\text{for any } z\notin\pi^{-1}(\pi(N(J\pi))),$$

where $N(J\pi)=\{J\pi=0\}$.

Proof of Corollary 1.2.4. We keep the notation from the previous proof. Keeping in mind the discussion from the begining of the proof of Theorem 1.2.1, observe that the reproducing property of the weighted Bergman kernel function implies that for any $f\in\mathbb{A}^2_\alpha(G)$ and $w\in D$ the following equalities hold

$$\begin{aligned}\langle\Gamma f, PK_D^{\alpha\circ\pi}(\cdot,w)\rangle_{\mathbb{A}^2_{\alpha\circ\pi}(D)} &= \langle\Gamma f, K_D^{\alpha\circ\pi}(\cdot,w)\rangle_{\mathbb{A}^2_{\alpha\circ\pi}(D)} = \Gamma f(w) = \frac{1}{\sqrt{m}}f(\pi(w))J\pi(w)\\ &= \langle f, K_G^\alpha(\cdot,\pi(w))\rangle_{\mathbb{A}^2_\alpha(G)}\frac{J\pi(w)}{\sqrt{m}} = \langle\Gamma f, \Gamma K_G^\alpha(\cdot,\pi(w))\rangle_{\mathbb{A}^2_{\alpha\circ\pi}(D)}\frac{J\pi(w)}{\sqrt{m}}.\end{aligned}$$

Consequently, from the Riesz Representation Theorem (uniqueness), applied to the space $\Gamma\mathbb{A}^2_\alpha(G)$, and the unitarity of Γ we get

$$\overline{J\pi(w)}K_G^\alpha(\pi(\cdot),\pi(w)) = \sqrt{m}\,(\Gamma^*\circ P)K_D^{\alpha\circ\pi}(\cdot,w)(\pi(\cdot)).$$

The last equality holds on $D\setminus\pi^{-1}(\pi(N(J\pi)))$ for arbitrary $w\in D$. But if we take $w\notin\pi^{-1}(\pi(N(J\pi)))$, then on the same set we have

$$K_G^\alpha(\pi(\cdot),\pi(w)) = (\Gamma^*\circ P)\frac{\sqrt{m}}{\overline{J\pi(w)}}K_D^{\alpha\circ\pi}(\cdot,w).$$

Unwinding the definitions of Γ^* and P produces the desired statement. □

Corollary 1.2.5. *Let* D *and* G *be domains in* $\mathbb{C}^n$ *and let* $\pi: D\to G$ *be a biholomorphic map. Then for* z, $w\in D$ *the following equality holds*

$$\overline{J\pi(z)}K_G(\pi(z),\pi(w))J\pi(w) = K_D(z,w).$$

In the next Section we will see how Theorem 1.2.1 and Corollary 1.2.4 work on an actual situation.

1.3. The Lu Qi-Keng problem on the tetrablock

Definition 1.3.1. Let

$$\varphi:\mathcal{R}_{II}\to\mathbb{C}^3,\ \varphi(z_{11},z_{22},z) := (z_{11},z_{22},z_{11}z_{22}-z^2),$$

where $\mathcal{R}_{II}$ denotes the classical Cartan domain of the second type (in $\mathbb{C}^3$), that is

$$\mathcal{R}_{II} = \{\widetilde{z}\in\mathcal{M}_{2\times2}(\mathbb{C}) : \widetilde{z}=\widetilde{z}^t,\ \|\widetilde{z}\|<1\},$$

where $\|\cdot\|$ is the operator norm and $\mathcal{M}_{2\times2}(\mathbb{C})$ denotes the space of 2×2 complex matrices (we identify a point $(z_{11},z_{22},z)\in\mathbb{C}^3$ with a 2×2 symmetric matrix $\begin{pmatrix} z_{11} & z\\ z & z_{22}\end{pmatrix}$). Then φ is a proper holomorphic map and $\varphi(\mathcal{R}_{II})=\mathbb{E}$ is a domain (see Proposition 1.4.1), called the *tetrablock*.

Recall that a domain D is *a Lu Qi-Keng domain* if its Bergman kernel function with weight $\alpha \equiv 1$ does not have zeros and is *not a Lu Qi-Keng domain* if it has.

As to the history of the Lu Qi Keng problem we refer to [**Boa1**]. There are many results on both, Lu Qi-Keng and not Lu Qi-Keng domains (cf. e.g. [**Boa2**], [**Yin-Zha**]) .

Recall that ([**Hua**] pg. 84)

$$K_{\mathcal{R}_{II}}(t,s) = \frac{1}{\mathrm{Vol}(\mathcal{R}_{II})}\Big(\det(I - t\overline{s})\Big)^{-3}, \quad \text{for } t,\, s \in \mathcal{R}_{II}.$$

Since every point in $\mathcal{R}_{II}$ can be carried by some automorphism of $\mathcal{R}_{II}$ into the origin (see [**Hua**] p. 84), we get $K_{\mathcal{R}_{II}} \neq 0$. Thus, $\mathcal{R}_{II}$ is a Lu Qi-Keng domain. Therefore, we have a proper holomorphic mapping $\varphi : \mathcal{R}_{II} \to \mathbb{E}$ of multiplicity 2 such that $\mathcal{R}_{II}$ is a Lu Qi-Keng domain whereas $\mathbb{E}$ is not a Lu Qi-Keng domain. Recall that another example of that type is $\{|z| + |w| < 1\} \ni (z,w) \to (z^2, w) \in \{|z|^{\frac{1}{2}} + |w| < 1\}$ (see [**Boa2**]). In our situation there is equality of holomorphically invariant distances in both domains and both domains are $\mathbb{C}$-convex (cf. [**E-K-Z**], [**Zwo**]) whereas in the example from [**Boa2**] it is not the case.(2)

Below we present two results which are consequences of Bell's transformation formula.

Corollary 1.3.2. *For any* $\widetilde{z} = (z_{11}, z_{22}, z)$, $\widetilde{w} = (w_{11}, w_{22}, w) \in \mathcal{R}_{II}$

$$J\varphi(\widetilde{z})\, K_{\mathbb{E}}(\varphi(\widetilde{z}), \varphi(\widetilde{w}))\, \overline{J\varphi(\widetilde{w})} = K_{\mathcal{R}_{II}}\Big((z_{11}, z_{22}, z), \widetilde{w}\Big) - K_{\mathcal{R}_{II}}\Big((z_{11}, z_{22}, -z), \widetilde{w}\Big).$$

A consequence of the last formula is the following:

Corollary 1.3.3. $\mathbb{E}$ *is not a Lu Qi-Keng domain.*

We set about achieving above Corollaries. We keep the notation from Section 1.2.

Proof of Corollary 1.3.2. Below we present how operators: Γ and P work- for a very special case, that is when $\pi = \varphi$, $\alpha = 1$, $D = \mathcal{R}_{II}$, $G = \mathbb{E}$. It is not necessary to do that to write down a formula for $K_{\mathbb{E}}$, but it is so simple in that case, that we think it is worth stating.

The range of the operator Γ is contained in the set of those maps whose coefficients at $z_{11}^k z_{22}^l z^{2n}$ in the Taylor expansion at the origin vanish for all k, l, n natural numbers. We showed that every function in $\Gamma\mathbb{A}^2(\mathbb{E})$ is of the form $J\pi \cdot h$ for some function h depending on z_{11}, z_{22}, z^2, but not necessarily conversely. The projection

$$P : \mathbb{A}^2(\mathcal{R}_{II}) \to \Gamma\mathbb{A}^2(\mathbb{E}),$$

acts as follows

$$P(f)(z_{11}, z_{22}, z) = \frac{1}{2}\Big(f(z_{11}, z_{22}, z) - f(z_{11}, z_{22}, -z)\Big), \quad f \in \mathbb{A}^2(\mathcal{R}_{II}), \qquad (z_{11}, z_{22}, z) \in \mathcal{R}_{II},$$

(2)For the definition of $\mathbb{C}$-convexity see Section 2.1.

and the adjoint

$$\Gamma^* g = \sqrt{2}\widetilde{\left(\frac{g}{\mathrm{J}\varphi}\right)}, \quad g \in \Gamma\mathbb{A}^2(\mathbb{E}).$$

From the proof of Collorary 1.2.4, we might write

$$K_{\mathbb{E}}(\varphi(\cdot), \varphi(w_{11}, w_{22,}w)) = (\Gamma^* \circ P)\frac{\sqrt{2}}{\overline{J\varphi(w_{11}, w_{22}, w)}} K_{\mathcal{R}_{II}}(\cdot, (w_{11}, w_{22}, w)),$$
$$\text{for } (w_{11}, w_{22}, w) \notin N(J\varphi),$$

and finally

$$K_{\mathbb{E}}(\varphi(z_{11}, z_{22}, z), \varphi(w_{11}, w_{22}, w)) =$$
$$\frac{K_{\mathcal{R}_{II}}\Big((z_{11}, z_{22}, z), (w_{11}, w_{22}, w)\Big) - K_{\mathcal{R}_{II}}\Big((z_{11}, z_{22}, -z), (w_{11}, w_{22}, w)\Big)}{J\varphi(z_{11}, z_{22}, z)\overline{J\varphi(w_{11}, w_{22}, w)}},$$

for (z_{11}, z_{22}, z), $(w_{11}, w_{22}, w) \notin N(J\varphi)$. □

Proof of Corollary 1.3.3. We examine the formula for Bergman kernel function for $\mathbb{E}$ for pair $\varphi(0,0,1)$, $\varphi(0,0,z)$ (note that the formula for the Bergman kernel function for $\mathcal{R}_{II}$ extends analytically to $\overline{\mathcal{R}_{II}} \times \mathcal{R}_{II}$). Calculation shows that $K_{\mathbb{E}}(\varphi(0,0,1), \varphi(0,0,z)) = \frac{\pi^3}{6}(3 + 10\overline{z}^2 + 3\overline{z}^4)(1 - \overline{z}^2)^{-6}$, $z \in \mathbb{D}$, and the last expression vanishes for $z_0^2 = -\frac{1}{3}$. Now the equality

$$K_{\mathbb{E}}(\varphi(0,0,1), \varphi(0,0,z_0)) = K_{\mathbb{E}}(\varphi(0,0,r), \varphi(0,0,\frac{1}{r}z_0)),$$

which holds for $0 < r < 1$ such that $\frac{z_0}{r} \in \mathbb{D}$, finishes the proof. □

1.4. Remark on proper holomorphic maps

Here we make an extension of Proposition 1 from [**Try1**]. Its proof is based on [**Rud1**, Proposition 2.1]. It gives the properness of a wide class of holomorphic mappings and the openness of their images, among others $\mathbb{G}_n$ and $\mathbb{E}$ (see remarks at the end of this Section).

Proposition 1.4.1. (see [**Try1**, Proposition1]) *Let D be a domain in $\mathbb{C}^n$ and let $k \in \mathbb{N} \cup \{0\}$. Let $f : D \to \mathbb{C}^n$ be a holomorphic map, and let $\varphi_j : D \to (0, +\infty)$, $j = 1, \ldots, k$ be continuous functions. Assume there exists a finite group of homeomorphic transformations $\mathcal{U}$ of D such that f is precisely $\mathcal{U}$-invariant, that is for z, $w \in D$ we have that $f(z) = f(w)$ if and only if $Uz = w$ for some $U \in \mathcal{U}$. Let $\mathbb{F} = \{(z_1, \ldots, z_k, z') \in \mathbb{C}^k \times D : |z_j| < \varphi_j(z'), j = 1, \ldots, k\}$ be a generalized Hartogs domain in $\mathbb{C}^{n+k}$ and $F = (P_{\mathbb{C}^k \times \{0\}^n}, f \circ P_{\{0\}^k \times \mathbb{C}^n}) : \mathbb{F} \to \mathbb{C}^{n+k}$, $F(z_1, \ldots, z_k, z') = (z_1, \ldots, z_k, f(z'))$, where P_A denotes the orthogonal projection onto A. Then $F(\mathbb{F})$ is a domain and $F : \mathbb{F} \to F(\mathbb{F})$ is a proper mapping.*

In Proposition 1.4.1 we only assumed that every $U \in \mathcal{U}$ is a homeomorphism but the equality $\pi \circ U = \pi$ easily implies that $\mathcal{U}$ actually is necessarily contained in the group of holomorphic automorphisms of D.

Proof of Proposition 1.4.1. Let $\{K_k\}_{k\in\mathbb{N}}$ be an increasing sequence of relatively compact domains of D exhausting D. Consider a new sequence $\{D_k := \bigcup_{U\in\mathcal{U}} U(K_k)\}_k$. Since $\mathcal{U}$ is finite, the set D_k is a relatively compact subset of D for every k. Moreover, there is some N such that for $k > N$ the set D_k is a domain. Certainly, this new sequence $\{D_k\}_k$ exhausts D. Fix $k > N$. As in [**Try1**] the idea is first to show the properness on some sequence of subdomains whose union gives the whole $\mathbb{F}$. For this, define a new sequence of subdomains $\mathbb{F}_k := \cup_{z'\in D_k}\mathbb{D}(\varphi_1(z')) \times \ldots \times \mathbb{D}(\varphi_k(z')) \times \{z'\}$. Then $\mathcal{U}_k = \{(P_{\mathbb{C}^k\times\{0\}^n}, U|_{D_k} \circ P_{\{0\}^k\times\mathbb{C}^n}) : U \in \mathcal{U}\}$ is a finite group of automorphisms of $\mathbb{F}_k$ and $F|_{\mathbb{F}_k} : \mathbb{F}_k \to \mathbb{C}^{n+k}$ is precisely $\mathcal{U}_k$-invariant. These two facts together with the properness of $U|_{D_k}$ as a selfmap of D_k for every $U \in \mathcal{U}$, imply that the intersection of the sets $F(\mathbb{F}_k)$ and $F(\partial\mathbb{F}_k)$ is empty. Let Ω_k be the component of $\mathbb{C}^{n+k} \setminus F(\partial\mathbb{F}_k)$ that contains $F(\mathbb{F}_k)$. Consequently, we get that $F(\partial\mathbb{F}_k) \subset \partial\Omega_k$. This implies that $F|_{\mathbb{F}_k} : \mathbb{F}_k \to \Omega_k$ is a proper map (here we used the fact that F is a holomorphic map on $\mathbb{F}_k$ which extends continuously to $\overline{\mathbb{F}_k}$).

Therefore, $\Omega_k = F(\mathbb{F}_k)$. Let $\Omega = \cup_k \Omega_k$. Evidently, $\Omega = F(\mathbb{F})$ is a domain in $\mathbb{C}^{n+k}$. The properness of F might be checked as follows. If $K \subset \Omega$ is compact, then $K \subset \Omega_k$ for some k. Hence $F^{-1}(K)$ is a compact subset of $\mathbb{F}_k$, and thus a compact subset of Ω. □

Remark 1.4.2. Let us consider the case when $k = 0$, $f = \varphi$ (defined in Section 1.2). Map φ is $\mathcal{U}_{\mathbb{E}} = \{\mathrm{Id}, \mathrm{diag}(1, 1, -1)\}$-invariant. What needs be to verified is only whether $\mathcal{U}_{\mathbb{E}}$ describes a subgroup of the group of automorphisms of $\mathcal{R}_{II}$. It can be derived by showing that the norm of matrix $\begin{pmatrix} z_{11} & z \\ z & z_{22} \end{pmatrix}$ (viewed as an operator on $\mathbb{C}^2$) equals the norm of the related matrix $\begin{pmatrix} z_{11} & -z \\ -z & z_{22} \end{pmatrix}$. But the norm of the matrix $\begin{pmatrix} z_{11} & -z \\ -z & z_{22} \end{pmatrix}$ equals

$$\sup_{a,b,c,d\in\mathbb{C}, |a|^2+|b|^2=1,\ |c|^2+|d|^2=1} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} z_{11} & -z \\ -z & z_{22} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix},$$

and clearly

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} z_{11} & -z \\ -z & z_{22} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a & -b \end{pmatrix} \begin{pmatrix} z_{11} & z \\ z & z_{22} \end{pmatrix} \begin{pmatrix} c \\ -d \end{pmatrix},$$

so the claim follows.

Remark 1.4.3. Let $k = 0$ and $f = s$ where s is the map given in Section 1.1. In that case the finite group of unitary transformations under which $s|_{\mathbb{D}^n}$ is precisely invariant is the group of permutations $\mathcal{S}_n$. Proposition 1.4.1 gives the proof of the fact that $s|_{\mathbb{D}^n}$ is a proper holomorphic mapping onto the image i.e. the symmetrized polydisc, and the symmetrized polydisc is open.

Remark 1.4.4. Proposition 1.4.1 might be applied to the pentablock as well. The pentablock is a new example of a domain which plays role in the μ-synthesis. However, in this thesis Remark 1.4.4 is the only place where we work with it. We refere the interested Reader to a recent paper by Kosiński [**Kos**] for more details on the pentablock. Among many results given in this paper there is the description [**Kos**, (1)] in a spirit to that given in the statement of Proposition 1.4.1 with $k = 1$. Consequently, the pentablock is actually a domain.

1.5. The Bergman distance on planar domains

In this Section we are largely interested in the Bergman distance. However, we also obtain comparisons of other holomorphically invariant distances.

The Section is organized as follows. First we introduced a few new notions reffering to metrics and distances (which the reader might find in e.g. [**Jar-Pfl2**]). After that we present the results from [**Nik-Try1**].

Definition 1.5.1. Let $\mathcal{G}$ denote the family of all domains in all $\mathbb{C}^n$'s. Let $s = (s_G)_{G \text{ is a domain}}$ be a system of functions $s_G : G \times G \to \mathbb{R}_{\geq 0}$ ($G \in \mathcal{G}$). We say that s is (*a holomorphically*) *contractible family of functions* if

$$s_{\mathbb{D}}(z,w) = c_{\mathbb{D}}(z,w) = k_{\mathbb{D}}(z,w) = \frac{1}{2}\log\frac{1+\left|\frac{z-w}{1-z\overline{w}}\right|}{1-\left|\frac{z-w}{1-z\overline{w}}\right|}, \quad \text{for } z,\ w \in \mathbb{D},$$

and

$$s_V(z,w) \geq s_G(F(z),F(w)), \quad \text{for } z, w \in V$$

for every $F \in \mathcal{O}(V,G)$ ($V,\ G \in \mathcal{G}$).

Let $\delta = (\delta_G)_{G\in\mathcal{G}}$ denote a system of pseudometrics $\delta_G : G \times \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ ($G \subset \mathbb{C}^n$), i.e. $\delta_G(a;\lambda X) = |\lambda|\delta_G(a;X)$, $\lambda \in \mathbb{C}$, $a \in G$, $X \in \mathbb{C}^n$. We say that δ is *a* (*holomorphically*) *contractible family of pseudometrics* if

$$\delta_{\mathbb{D}}(z;X) = \frac{|X|}{1-|z|^2},\ z \in \mathbb{D},\ X \in \mathbb{C},$$

and if for any $F \in \mathcal{O}(V,G)$ ($V,\ G \in \mathcal{G}$)

$$\delta_V(z;X) \geq \delta_G(F(z);F'(z)X),\ z \in V \subset \mathbb{C}^n,\ X \in \mathbb{C}^n.$$

Now we show how to construct a contractible family of pseudodistances from a given system of upper semicontinous (!) contractible family of pseudometrics (cf. [**Jar-Pfl2**, pg. 140-142]). Let $\alpha : [0,1] \to G$ be any $\mathcal{C}^1$-piecewise curve, $G \in \mathcal{G}$. Put

$$L_{\delta_G}(\alpha) := \int_0^1 \delta_G(\alpha(t);\alpha'(t))dt.$$

The number $L_{\delta_G}(\alpha)$ is called *the δ-length of α*. Define a pseudodistance on G in the following way

$$\Big(\int \delta_G\Big)(z', z") := \inf\{L_{\delta_G}(\alpha) : \alpha : [0,1] \to G\ \mathcal{C}^1 - \text{piecewise}, \qquad \alpha(0) = z',\ \alpha(z'') = 1\},\ z', z'' \in G.$$

We say that $\int \delta_G$ is *the integrated form of δ_G*.

In this thesis we meet many times the above construction. The first example is given in the next definition.

Let D be a bounded domain in $\mathbb{C}^n$.

Definition 1.5.2. *The Bergman distance* b_D *of* D is the integrated form of the Bergman metric β_D, i.e.

$$b_D(z,w) := \Big(\int \beta_D\Big)(z,w), \quad z,w \in D.$$

Definition 1.5.3. Denote by c_D and l_D *the Carathéodory distance* and *the Lempert function* of D, respectively:

$$c_D(z,w) := \sup\{\tanh^{-1}|f(w)| : f \in \mathcal{O}(D,\mathbb{D}), \text{ with } f(z)=0\},$$

$$l_D(z,w) := \inf\{\tanh^{-1}|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D},D) \text{ with } \varphi(0)=z, \varphi(\alpha)=w\},\ z,\ w \in D,$$

The Kobayashi distance k_D is the largest pseudodistance not exceeding l_D.

Recall that the Kobayashi distance is the integrated form of *the Kobayashi metric* κ_D defined by the formula

$$\kappa_D(z;X) := \inf\Big\{|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D},D) \text{ with } \varphi(0)=z,\ \alpha\varphi'(0)=X\Big\},\ z \in D,\ X \in \mathbb{C}^n$$

(cf. e.g. [**Jar-Pfl2**, Theorem 3.6.4]).

In Section 2.3 we work with the infinitesimal version of the Carathéodory distance. To have all notions in one place we define it here. *The Carathéodory metric* of a domain G, denoted γ_G, is defined as follows

$$\gamma_G(\zeta;X) := \sup\Big\{|f'(\zeta)X| : f \in \mathcal{O}(G,\mathbb{D})\Big\}, \quad \zeta \in G \subset \mathbb{C}^n,\ X \in \mathbb{C}^n.$$

Recall that

• $\kappa = (\kappa_G)_{G\in\mathcal{G}}$ and $\gamma = (\gamma_G)_{G\in\mathcal{G}}$ are examples of holomorphically contractible families of pseudometrics (cf. [**Jar-Pfl2**]);

• $\beta = (\beta_G)_{G\in\mathcal{G}}$ is a family of pseudometrics which is not holomorphically contractible (cf. [**Pfl-Zwo**]);

• $c = (c_G)_{G\in\mathcal{G}}$ and $k = (k_G)_{G\in\mathcal{G}}$ are holomorphically contractible families of pseudodistances (cf. [**Jar-Pfl2**]).

An immediate consequence of Definition 1.5.3 is

$$c_D \le k_D \le l_D.$$

Moreover, $k_D = l_D$ for any planar domain D (cf. [**Jar-Pfl2**, Remark 3.3.8(e)]).

Among properties of the objects introduced in Definition 1.5.3 is *the product property.* Namely

$$s_{D_1\times\ldots\times D_n}\big((z_1,\ldots,z_n),(w_1,\ldots,w_n)\big) = \max\nolimits_{1\le j\le n} s_{D_j}(z_j,w_j),\ z_j,\ w_j \in D_j, \quad (1.5.1)$$

where $s_D = c_D,\ l_D$ or k_D (cf. [**Jar-Pfl2**]).

The motivation for our next result Proposition 1.5.8 was the following simple observation for the unit disc.

Lemma 1.5.4. *For any z, $w \in \mathbb{D}$ the following holds*

$$\log\Big(1+\frac{|z-w|}{\sqrt{(1-|z|^2)(1-|w|^2)}}\Big) \le \frac{b_{\mathbb{D}}(z,w)}{\sqrt{2}}$$
$$= c_{\mathbb{D}}(z,w) = k_{\mathbb{D}}(z,w) \le \log\Big(1+\frac{2|z-w|}{\sqrt{(1-|z|^2)(1-|w|^2)}}\Big). \quad (1.5.2)$$

PROOF. We have

$$\sqrt{2}b_{\mathbb{D}}(z,w) = 2k_{\mathbb{D}}(z,w) = \log\frac{1+\left|\frac{z-w}{1-\bar{z}w}\right|}{1-\left|\frac{z-w}{1-\bar{z}w}\right|} =$$

$$\log\left(1+\frac{2|z-w|}{|1-\bar{z}w|-|z-w|}\right) = \log\left(1+2|z-w|\frac{|1-\bar{z}w|+|z-w|}{(1-|z|^2)(1-|w|^2)}\right).$$

Observe that

$$|1-\bar{z}w|^2 = (1-|z|^2)(1-|w|^2)+|z-w|^2, \quad (1.5.3)$$

and hence

$$\sqrt{(1-|z|^2)(1-|w|^2)} \le |1-\bar{z}w| \le \sqrt{(1-|z|^2)(1-|w|^2)}+|z-w|. \quad (1.5.4)$$

Elementary calculation together with (1.5.3) and (1.5.4) shows that

$$\log\left(1+\frac{|z-w|}{\sqrt{(1-|z|^2)(1-|w|^2)}}\right)$$
$$\le \frac{1}{2}\log\left(1+2|z-w|\frac{|1-\bar{z}w|+|z-w|}{(1-|z|^2)(1-|w|^2)}\right)$$
$$\le \log\left(1+\frac{2|z-w|}{\sqrt{(1-|z|^2)(1-|w|^2)}}\right),$$

and in this way we finish the proof. □

Observe that the estimate (1.5.3) is very accurate and seems to be quite natural. Indeed, the difference between the right and the left side is at most log 2.

Proposition 1.5.8 is an attempt to extend Lemma 1.5.4 on Dini-smooth domains.

Definition 1.5.5. (see e.g. [**Pom1**, Chapter 3])

Let $\varsigma : [a,b] \to \mathbb{C}$ be continuous. $C := \varsigma^* :=$ graph ς is called *a curve* (*parametrized by* ς). A curve C is called closed if $\varsigma(a) = \varsigma(b)$.

We call C *a Jordan arc* if C is a curve with some ς injective. A curve C parametrized by $\varsigma : [a,b] \to \mathbb{C}$ is *a Jordan curve* if $\varsigma|_{[a,b)}$ is injective and ς is closed.

Recall that if J is a Jordan curve in $\hat{\mathbb{C}}$, then $\hat{\mathbb{C}} \setminus J$ has exactly two components G_0 and G_1, and these satisfy $\partial G_0 = \partial G_1 = J$ (the Jordan Curve Theorem, see c.f. [**Pom1**, Chapter 1]). If $J \subset \mathbb{C}$ the bounded component of $\mathbb{C} \setminus J$ will be called *the inner domain of* J.

A domain D bounded by a Jordan curve J is called *a Jordan domain*.

Let ϑ be uniformly coninuous function on the connected set $A \subset \mathbb{C}$. Its *modulus of continuity* is defined by

$$\omega(t) := \sup\{|\vartheta(t_1) - \vartheta(t_2)| : |t_1 - t_2| \leq t\}, \quad t > 0.$$

The function ϑ is called *Dini-continuous* if

$$\int_0^\delta \frac{\omega(t)}{t} dt < \infty, \quad \text{for some } \delta > 0, \tag{1.5.5}$$

for some $\delta > 0$.

We say that the curve C is *Dini-smooth* if it has a $\mathcal{C}^1$ parametrization $\gamma : [-\pi, \pi] \to \mathbb{C}$ such that γ' is Dini-continuous and $\gamma' \neq 0$.

Bounded domain $D \subset \mathbb{C}$ is called *Dini-smooth* if $\partial D = \gamma^*$, where $\gamma : [-\pi, \pi] \to \mathbb{C}$ is a Dini-smooth Jordan curve such that $\lim_{t\to-\pi} \gamma'(t) = \lim_{t\to\pi} \gamma'(t)$.

Further, motivated by Definition 1.5.5, we say that a planar domain D is *Dini smooth* at a ($\in \partial D$) if there exist a neighborhood U of a and a Dini-smooth Jordan arc γ such that $\partial D \cap U = \gamma^*$. Clearly, every Dini-smooth domain is Dini-smooth at every boundary point. Conversly, by the compactness argument, if D is a bounded domain in $\mathbb{C}$ and D is Dini-smooth at every boundary point, then D is Dini-smooth.

Let C be a Jordan curve in $\mathbb{C}$. Assume that C has $\mathcal{C}^{1,\alpha}$ parametrization $\gamma : [-\pi, \pi] \to \mathbb{C}$ ($\alpha > 0$), $\lim_{t\to\pi} \gamma'(t) = \lim_{t\to-\pi} \gamma'(t)$. Then, clearly the condition (1.5.5) holds for γ. Thus the Dini-smoothness condition is weaker than $\mathcal{C}^2$. (Recall that $\mathcal{C}^{k,\alpha}(D)$ is the space of all k times continuously differentiable functions $f : D \to \mathbb{C}$ such that $|f^{(k)}(x) - f^{(k)}(y)| \leq M|x - y|^\alpha$ for some $M > 0$ and all x, y in an open set $D \subset \mathbb{R}^l$, $k, l \in \mathbb{N}$, $\alpha > 0$). Moreover, by the example after Proposition 1.5.10 it is stronger than $\mathcal{C}^1$.

The condition (1.5.5) does not appear explicitly in the proof of Proposition 1.5.8 and apparently looks artificial. But this is misleading. Its importance is well known in the theory of conformal mappings. Namely

Theorem 1.5.6. (**Warschawski Theorem** cf. [**Pom1**, Theorem 3.5])

Let F maps $\mathbb{D}$ conformally $\mathbb{D}$ onto the inner domain of the Dini-smooth Jordan curve J. Then F' extends continuously to $\overline{\mathbb{D}}$ and

$$\lim_{z\to w} \frac{F(z) - F(w)}{z - w} = F'(w) \neq 0, \; z, w \in \overline{\mathbb{D}}.$$

Theorem 1.5.6 implies that

$$d_{\mathbb{D}}(z) \sim d_D(F(z)) \tag{1.5.6}$$

for $z \in D$ sufficiently close to a if D is Dini-smooth at a, see the simplification below. If a domain D is Dini-smooth at a, then by the localization by Forstneric and Rossay (see Theorem 1.5.14) the distances b_D and $b_{\mathbb{D}}$ are equal near a modulo some conformal map (we precise it later). This fact is cruical in the study of the Bergman and the Kobayashi distances on Dini-smooth domains.

Recall that Nikolov in [**Nik1**] pondered the Carathéodory and the Kobayashi distances on Dini-smooth domains.

Proposition 1.5.7. ([**Nik1**, Proposition 8]) *Let D be a Dini-smooth bounded planar domain. Then there exists a constant $c_1 > 1$ such that*

$$\log\Big(1+\frac{|z-w|}{c_1\sqrt{d_D(z)d_D(w)}}+\frac{|z-w|^2}{c_1 d_D(z)d_D(w)}\Big)\le s_D(z,w)$$
$$\le \log\Big(1+\frac{c_1|z-w|}{\sqrt{d_D(z)d_D(w)}}+\frac{c_1|z-w|^2}{d_D(z)d_D(w)}\Big),\quad z,\,w\in D,\quad (1.5.7)$$

where $s_D(z,w)=2c_D(z,w)$ or $s_D(z,w)=2k_D(z,w)$.

Trying to repeat the reasoning given in [**Nik1**] for the Bergman distance we find many obstacles. It happens because there is a significant difference between κ_D and β_D, i.e. the Bergman metric is not holomorphically contractive already in dimension one (!) (cf. [**Jar-Pfl2**, pg. 187] or [**Pfl-Zwo**]). However, we have

Proposition 1.5.8. ([**Nik-Try2**, Proposition 1]) *Led D be a Dini-smooth bounded planar domain. Then there exists a constant $c > 1$ such that*

$$\sqrt{2}\log\Big(1+\frac{|z-w|}{c\sqrt{d_D(z)d_D(w)}}\Big)\le b_D(z,w)$$
$$\le\sqrt{2}\log\Big(1+\frac{c|z-w|}{\sqrt{d_D(z)d_D(w)}}\Big),\quad z,\,w\in D.\quad (1.5.8)$$

Consequently, we obtain

Corollary 1.5.9. *If D is a Dini-smooth bounded planar domain, then the differences $b_D-\sqrt{2}c_D$ and $b_D-\sqrt{2}k_D$ are bounded.*

Let us try to understand the essence of: (1.5.7), (1.5.8). Observe that

$$\log(1+x)\sim\log x\quad\text{if}\quad x>>1,$$
$$\log(1+x)\sim x\quad\text{if}\quad 0<x<<1.$$

By this and the continuity of the Bergman metric, we see that Proposition 1.5.8 is equivalent to

Proposition 1.5.10. *Let D be a Dini-smooth bounded planar domain. There exists a constant $c > 1$ such that:*

- *if $|z-w|^2 > d_D(z)d_D(w)$ then*

$$\log\frac{|z-w|^2}{d_D(z)d_D(w)}-c<\sqrt{2}b_D(z,w)<\log\frac{|z-w|^2}{d_D(z)d_D(w)}+c;$$

- *if $|z-w|^2\le d_D(z)d_D(w)$ then*

$$\frac{|z-w|}{c\sqrt{d_D(z)d_D(w)}}\le b_D(z,w)\le\frac{c|z-w|}{\sqrt{d_D(z)d_D(w)}}.$$

If the regularity condition is missing, then there is no constant such that the upper bound in (1.5.8) or (1.5.7) holds. Indeed, let $D\subset\mathbb{C}$ be the image of $\mathbb{D}$ under the map $z \to 2z + (1-z)\log(1-z)$. Then D is a $\mathcal{C}^1$-smooth bounded domain (cf. [**Pom1**, pg. 46]) and

$$\lim_{\mathbb{R}\ni w\to 2^-} \frac{1-\tanh l_D(0,w)}{d_D(w)} = 0$$

(see [**N-P-T**, Example 2]).

First we want to show that some simplifications in Proposition 1.5.8 are possible. For this let us quote the following results.

Proposition 1.5.11. ([**Nik1**, Proposition 5]) *Assume that D is a Dini-smooth domain. For every point $p \in \partial D$ and any compact subset K of D, there exist a neighborhood V of p, and a constant $c > 0$ such that*

$$|2s_D(z,w) + \log d_D(w)| \le c, \quad z \in K,\ w \in D \cap V,$$

where $s_D = c_D$, $s_D = k_D$, or $s_D = b_D/\sqrt{2}$.

Corollary 1.5.12. ([**Nik1**, Corollary 6]) *Assume that D is a Dini-smooth, bounded domain. Let p, q be different boundary points of D. If $s_D = k_D$ or $s_D = b_D/\sqrt{2}$, then the function*

$$2s_D(z,w) + \log d_D(z) + \log d_D(w)$$

is bounded for z near q and w near p.

In the light of Proposition 1.5.11 and Corollary 1.5.12 distances $b_D/\sqrt{2}$ and k_D have the same behavior when points z and w are far away. More precisely, for every $\epsilon > 0$ there exist: $\delta > 0$ and a positive constant c_1 such that for every z, $w \in D$ if $d_D(z) < \epsilon$ and $d_D(w) > \delta$, then

$$|2k_D(z,w) + \log d_D(z) + \log d_D(w)| < c_1 \tag{1.5.9}$$

and

$$|\sqrt{2}b_D(z,w) + \log d_D(z) + \log d_D(w)| < c_1, \tag{1.5.10}$$

Thus, to derive Proposition 1.5.8 it remains to deal with the case when **points** z **and** w **are near the same boundary point p**. Further discussion concerns only this case.

We will make one more simplification. For this we need some additional notations.

Definition 1.5.13. (cf. [**Jar-Pfl3**, Chapter 2.2]) Let Ω be a domain in $\mathbb{C}^n$. Fix an open set U that contains the closure of Ω. Let $\rho : U \to \mathbb{R}$ be a function.

We say that ρ is *a defining function for* Ω if ρ is $\mathcal{C}^1$ and if ρ satisfies the following conditions:

(1) $\Omega = \{x \in U : \rho(z) < 0\}$,
(2) $\partial\Omega = \{z \in U : \rho(z) = 0\}$,
(3) $\rho' \ne 0$ on U.

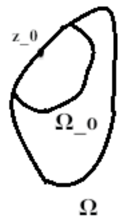

FIGURE 1.

Further, we say that a domain Ω is *strictly* or *strongly pseudoconvex* if Ω has $\mathcal{C}^2$ smooth defining function ρ satisfying

$$\mathcal{L}_\rho(a;X) = \sum_{k,l=1}^{n} \frac{\partial^2\rho(a)}{\partial z_k \partial \overline{z}_l} X_k \overline{X}_l > 0, \quad a \in \partial\Omega,\ X \in \mathbb{C}^n \setminus \{0\}. \tag{1.5.11}$$

The function $\mathcal{L}_\rho$ is known in the literature as *the Levi form of* ρ. If the condition (1.5.11) holds only for points a near a_0 ($\in \partial D$), then we say that Ω *is strictly pseudoconvex at* a_0.

As we mentioned one of the tools that we use in the proof of Proposition 1.5.8 is the localization of the Kobayashi metric. The localization that we have in mind is the content of the next theorem.

Theorem 1.5.14. ([**For-Ros**, Theorem 2.1]) *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that Ω is strictly pseudoconvex at a point z_0 ($\in \partial\Omega$). Let $\Omega_0 \subset \Omega$ be a domain such that $z_0 \in \operatorname{int}_{\partial\Omega} \partial\Omega_0$ (see Figure 1). Then there exists a neighbborhood U of z_0 and a constant $c > 0$ such that for any point $z \in \Omega_0 \cap U$, any vector $X \in \mathbb{C}^n$ the following relation between κ_Ω and κ_{Ω_0} holds*

$$\kappa_\Omega(z;X) \geq \Big(1 - c\, d_\Omega(z)\Big)\kappa_{\Omega_0}(z;X). \tag{1.5.12}$$

However, for our modest purposes the weak version of Theorem 1.5.14 is enough. Indeed, taking into account (1.5.7) we will try to replace in Proposition 1.5.8 the domain D by the image of some conformal map. We might find a Dini-smooth Jordan curve $\zeta : [-\pi,\pi] \to \mathbb{C}$ such that $\zeta^* \cap U = \partial D \cap U$ for some neighborhood U of p and $D \subset G = \zeta_{\text{ext}}(:= \mathbb{C}\backslash$(a bounded domain whose boundary is equal to ζ)). (see Figure 2). Take a point $a \notin \overline{G}$ and consider the union G_e of 0 and the image of G under the map $\varphi : z \to (z-a)^{-1}$. There exists a conformal map $\psi : G_e \to \mathbb{D}$. By Warschawski Theorem ψ extends to a $\mathcal{C}^1$ diffeomorphism from $\overline{G}_e$ to $\overline{\mathbb{D}}$. Furthermore, after rotation we may assume that $p = 1$. Thus, there exists a positive $r_0 \in (0,1)$ such that $\mathbb{D} \cap \mathbb{D}(1,r) \subset D$. Notice that for every $N \in \mathbb{N}$, for every $0 < s_1 < s_2 < \ldots < s_N < r_0$ we might enlarge a little the sets $\{\mathbb{D} \cap \mathbb{D}(1,s_j)\}_{j=1}^N$ to the smooth domains $\{E_{s_j}\}$ in such way that

$$E_{s_1} \subset E_{s_2} \ldots \subset E_{s_N} \subset D$$

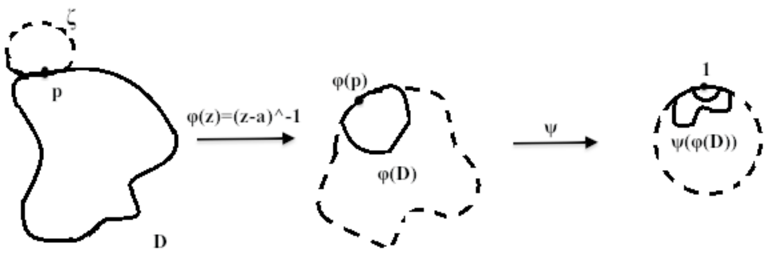

FIGURE 2.

and

$$\inf\Big\{|x-y| : x \in \partial E_{s_j} \cap \mathbb{D},\ y \in \partial E_{s_l} \cap \mathbb{D},\ 1 \le j < l \le N\Big\} > 0.$$

Consequently, to finish the proof of Proposition 1.5.8 we might assume that **the domain D is a subset of the unit disc such that $\mathbb{D}(1, r_0) \cap \mathbb{D} \subset D$ for some $r_0 > 0$, and that points z, w are close to the point $p = 1$.**

In the situation just described the Köbe Quarter Theorem provides an easy and short proof of Theorem 1.5.14. For the sake of completeness we quote the Köbe Quarter Theorem and after this present the proof of Theorem 1.5.14.

Theorem 1.5.15. Köbe Quarter Theorem (cf. [**Pom2**, pg. 21-22])

The image of an injective analytic function $f : \mathbb{D} \to \mathbb{C}$ from the unit disk $\mathbb{D}$ onto a subset of the complex plane contains the disc whose center is $f(0)$ and whose radius is $|f'(0)|/4$.

PROOF OF THEOREM 1.5.14 FOR $\Omega = \mathbb{D}, \Omega_0 = D$. Fix $z \in E_r$. We must show that there exists a positive constant $c > 0$ such that the estimate (**??**) holds and that c does not depend on the choice of z. By the Riemann Mapping Theorem there exists a conformal map $\phi : \mathbb{D} \to E_r$ satisfying $\phi(z) = z$. By Theorem 1.5.15

$$\mathbb{D}\Big(z, \frac{1}{4}|(\phi \circ \frac{\cdot + z}{1 + \cdot\ \overline{z}})'(0)|\Big) \subset E_r.$$

But $|(\phi \circ \frac{\cdot+z}{1+\cdot\ \overline{z}})'(0)| = |\phi'(z)|(1 - |z|^2)$, and now because of Warschawski's Theorem it remains to use a Dini-smoothness of E_r. □

The last step before we give the proof of Proposition 1.5.8 is the Balogh-Bonk's Theorem.

Assume that $\Omega \subset \mathbb{C}^n$, $n \ge 2$ is strictly pseudoconvex and ρ is a defining function of Ω with $|\rho'| = 1$ on $\partial\Omega$ (such ρ exists, cf. [**Jar-Pfl3**, Chapter 2] or [**Kra-Par**, Chapter 1]). Take arbitrary $x \in \Omega$. If x is sufficenty close to $\partial\Omega$, then there exists only one point $\pi(x) \in \partial\Omega$ such that $d_\Omega(x) = |x - \pi(x)|$ (cf. [**Jar-Pfl3**, Chapter 2] or [**Kra-Par**, Chapter 1]). Thus we have the mapping

$$\pi : (\text{neighborhood of } \partial\Omega) \cap \Omega \to \partial\Omega.$$

Assume x is so near. For $Z \in \mathbb{C}^n$ we put $Z_N := \langle Z, \rho'(\pi(x))\rangle \rho'(\pi(x))$, $Z_H := Z - Z_N$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{C}^n$.

Call a piecewise $\mathcal{C}^1$ smooth curve $\alpha : [0,1] \to \partial\Omega$ *horizontal* if for $t \in [0,1]$ for which $\alpha'(t)$ exists we have $\alpha'(t) = \alpha'(t)_H$. By the strict pseudoconvexity of Ω its boundary $\partial\Omega$ is connected. Even more is true, any two points p, $q \in \partial\Omega$ can be joined by a horizontal curve α (cf. [**Bal-Bon**, pg. 513]). For p, $q \in \partial\Omega$ let

$$d_H(p,q) = \inf\{L_{\mathcal{L}_\rho^{1/2}}(\alpha) : \alpha : [0,1] \to \partial\Omega \text{ is a horizontal curve with } \alpha(0) = p,\ \alpha(1) = q\},$$

where $L_{\mathcal{L}_\rho^{1/2}}(\alpha)$ is define as δ-lenght of α (see pg. 24). The d_H is called *the horizontal* or *Carnot-Carathéodory metric* on $\partial\Omega$ (a recent account on the subject can be found in [**Bella**] and [**Gro2**]). Finally we define

$$g(x,y) := 2\,\log\left[\frac{d_H(\pi(x),\pi(y)) + \max\{d_\Omega(x), d_\Omega(y)\}}{\sqrt{d_\Omega(x)d_\Omega(y)}}\right] \tag{1.5.13}$$

for all x, y from sufficiently small neighborhood of $\partial\Omega$.

The expression g has the advantage that it is a distance if we restrict it to a sufficently small neighborhood of $\partial\Omega$. (All the facts just cited relating to the objects introduced after the proof of Theorem 15.14 the Reader might find in [**Bal-Bon**].)

Theorem 1.5.16. ([**Bal-Bon**, Theorem 1.1]) *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n, n \geq 2$ with a defining function ρ. Assume that F is a metric on Ω, i.e. $F : \Omega \times \mathbb{C}^n \to \mathbb{R}_{\geq 0}$, $F(a, \lambda X) = |\lambda| F(a,X)$, $a \in \Omega$, $X \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, with the following property. There exist constants $\epsilon_0 > 0$, $s > 0$, $C_1 > 0$, $C_2 \geq 1$ such that for all x such that $d_\Omega(x) < \epsilon_0$ and $Z \in \mathbb{C}^n$ we have*

$$\begin{aligned}(1 - C_1\, d_\Omega(x)^s)\Big(\frac{|Z_N|^2}{4\, d_\Omega(x)^2} + \frac{1}{C_2}\,\frac{\mathcal{L}_\rho(\pi(x); Z_H)}{d_\Omega(x)}\Big)^{1/2} &\leq F(x;Z)\\ &\leq (1 + C_1\, d_\Omega(x)^s)\Big(\frac{|Z_N|^2}{4\, d_\Omega(x)^2} + C_2\,\frac{\mathcal{L}_\rho(\pi(x); Z_H)}{d_\Omega(x)}\Big)^{1/2}\end{aligned} \tag{1.5.14}$$

If d_F is the distance function associated with F, i.e. d_F is the integrated form of F, $d_F = (\int F)$ (see the construction on pg. 24), then there exists a constant $C \geq 0$ such that for all x, $y \in \Omega$

$$g(x,y) - C \leq d_F(x,y) \leq g(x,y) + C.$$

Theorem 1.5.17. ([**Bal-Bon**, Proposition 1.2]) *If Ω is strictly pseudoconvex in $\mathbb{C}^n$, $n \geq 2$, then the Kobayashi metric κ_Ω satisfies the assumptions of Theorem 1.5.16 near the boundary of Ω*

Observe that g does not depend on F (!). It is only important that the metric satisfies the condition of that kind as (1.5.14) near $\partial\Omega$.

Below we apply Theorem 1.5.16 for the 'modified' Kobayashi metric on the unit ball $\mathbb{B}_2$ in $\mathbb{C}^2$. However, in Section 2.2 we make use of Theorem 1.5.16 in its full generality.

PROOF OF PROPOSITION 1.5.8. After the preparations preceding the proof, the situation profiles as follows. There exists $0 < r_0 << 1$ such that for every $N \in \mathbb{N}$, $0 < r_1 < r_2 < \dots r_N < r_0$ there exists the sets $\{E_{s_j}\}_{j=1}^N$ such that

$$\mathbb{D}(1, s_j) \cap \mathbb{D} \subset E_{s_j},\ j = 1, \dots, N,$$
$$E_{s_1} \subset E_{s_2} \dots \subset E_{s_N} \subset D,$$
$$\inf\Big\{|x - y| : x \in \partial E_{s_j} \cap \mathbb{D},\ y \in \partial E_{s_l} \cap \mathbb{D},\ 1 \le j < l \le N\Big\} > 0.$$

Recall that we investigate b_D near a point 1.

It is enough to find a constant $c > 1$ such that the respective estimates hold for $b_D(z_n, w_n)$ for every sequences (z_n), $(w_n) \subset D$ such that $z_n \to 1$ and $w_n \to 1$ for any n.

For a planar domain Ω set $\beta_\Omega(z) := \beta_\Omega(z; 1)$, $M_\Omega(z) := M_\Omega(z; 1)$ and $\kappa_\Omega(z) := \kappa_\Omega(z; 1)$ for a point $z \in \Omega$.

Then by (1.1.4) we might write the following

$$\sqrt{2}\frac{\kappa_{\mathbb{D}}^2(z)}{\kappa_{E_r}(z)} = \frac{M_{\mathbb{D}}(z)}{\sqrt{K_{E_r}(z)}} \le \beta_D(z) \le \frac{M_{E_r}(z)}{\sqrt{K_{\mathbb{D}}(z)}} = \sqrt{2}\frac{\kappa_{E_r}^2(z)}{\kappa_{\mathbb{D}}(z)}, \quad z \in E_r \tag{1.5.15}$$

(the both equalities hold because E_r is a simply connected domain, here the smoothness of D is not required).

Fix an $r_1 \in (0, r_0)$. The localization of the Kobayashi metric (Theorem 1.5.14) implies that

$$\kappa_{\mathbb{D}}(z) \ge (1 - c_2\, d_{\mathbb{D}}(z))\kappa_{E_r}(z), \quad z \in E_{r_1}, \tag{1.5.16}$$

for some constant $c_2 > 0$. Then (1.5.15) and (1.5.16) imply that there exists a $r_2 \in (0, r_1]$ with $3c_2 r_2 \le 1$ such that

$$\sqrt{2}\big(1 - c_2\, d_{\mathbb{D}}(z)\big)\kappa_{\mathbb{D}}(z) \le \beta_D(z) \le \sqrt{2}\big(1 + \frac{5}{2}c_2 d_{\mathbb{D}}(z)\big)\kappa_{\mathbb{D}}(z), \quad z \in E_{r_2}$$

Since $\kappa_{\mathbb{D}}(z) = \frac{\beta_{\mathbb{D}}(z)}{\sqrt{2}} = \frac{1}{1-|z|^2}$, it follows for $c_3 = \frac{\sqrt{2}}{2}c_2$ that

$$\frac{\beta_{\mathbb{D}}(z)}{3} < \beta_{\mathbb{D}}(z) - 2c_3 < \beta_D(z) < \beta_{\mathbb{D}}(z) + 5c_3, \quad z \in E_{r_2}. \tag{1.5.17}$$

We may assume that $z_n, w_n \in E_{r_3}$, where $r_3 \in (0, r_2)$ is such that if α_n is the shorter arc with endpoints z_n and w_n of the circle through z_n and w_n which is orthogonal to the unit circle, then $\alpha_n \subset E_{r_2}$. Hence

$$b_D(z_n, w_n) < \int_{\alpha_n} \left(\frac{\sqrt{2}}{1 - |z|^2} + 5c_3\right) dl$$
$$= b_{\mathbb{D}}(z_n, w_n) + 5c_3\, L_{|\ |}(\alpha_n) < b_{\mathbb{D}}(z_n, w_n) + 10c_3|z_n - w_n|$$

for any n. Recall that for every curve $\alpha : [0, 1] \to \mathbb{R}^2$, we may define the $|\ |$-length as follows

$$L_{|\ |}(\alpha) := \sup\Big\{\sum_{j=1}^N |\alpha(t_{j-1}) - \alpha(t_j)| : N \in \mathbb{N},\ 0 = t_0 < t_1 < \dots < t_N = 1\Big\}.$$

The above equality follows from the description of the shortest curves with respect to the Bergman distance on $\mathbb{D}$ (cf. [**Kra1**, Section 1.1] and [**Jar-Pfl2**, Chapter

1, Chapter 6]). To get the second inequality we applied an elementary inequality $1 \le \frac{x}{\sin x} < 2$ for $x \in (0, \frac{\pi}{2})$.

Now, using Lemma 1.5.4 and the inequality

$$d_{\mathbb{D}}(z) \ge d_D(z), \quad z \in D,$$

it is easy to find a constant $c > 1$ such that the upper estimate for $b_D(z_n, w_n)$ in Proposition 1.5.8 holds for any n.

It is left to manage with the lower estimate. Shrinking r_3 (if necessary), we may assume that

$$d_{\mathbb{D}}(z) = d_D(z), \quad z \in E_{r_3}. \tag{1.5.18}$$

Consider the set A of all n for which there exists a smooth curve $\gamma_n : [0,1] \to D$ such that $\gamma_n(0) = z_n$, $\gamma_n(1) = w_n$, $\gamma_n\big((0,1)\big) \not\subset E_{r_2}$ and

$$b_D(z_n, w_n) + |z_n - w_n| > \int_0^1 \beta_D(\gamma_n(t); \gamma_n'(t))dt.$$

For any $n \in A$ we may find a number $t_n \in (0,1)$ such that $|u_n - 1| = r_2$, where $u_n = \gamma(t_n)$. By (1.5.10), there exists a constant $c_4 > 0$, which does not depend on $n \in A$, such that

$$\begin{aligned} b_D(z_n, w_n) + |z_n - w_n| &> b_D(z_n, u_n) + b_D(u_n, w_n) \\ &> -\frac{\log d_D(z_n)}{\sqrt{2}} - \frac{\log d_D(w_n)}{\sqrt{2}} - c_4. \end{aligned}$$

This inequality easily provides a constant $c > 1$ for which the lower estimate for $b_D(z_n, w_n)$ in Proposition 1.5.8 holds for any $n \in A$.

Let now $n \notin A$. Then, using (1.5.17) and the formula for the Kobayashi metric for the unit ball (cf. [**Jar-Pfl2**, Corollary 2.3.5])

$$\kappa^2_{\mathbb{B}_2}(w; Y) = \frac{|Y|^2}{1 - |w|^2} + \frac{|\langle w, Y\rangle|^2}{(1 - |w|^2)^2}, \quad w \in \mathbb{B}_2,\ Y \in \mathbb{C}^2,$$

we get that

$$b_D(z_n, w_n) + |z_n - w_n| \ge \sqrt{2}\hat{k}_{\mathbb{B}_2}\big((z_n, 0), (w_n, 0)\big),$$

where $\hat{k}_{\mathbb{B}_2}$ is the pseudodistance arising from the Finsler pseudometric $\hat{\kappa}_{\mathbb{B}_2}(w; Y) = (\kappa_{\mathbb{B}_2}(w; Y) - 2c_2||Y||)^+$ (i.e. its integrated form). Applying [**Bal-Bon**, Theorem 1.1] to $\kappa_{\mathbb{B}_2}$ and $\hat{\kappa}_{\mathbb{B}_2}$, we may find a constant $c_5 > 0$ such that $0 < k_{\mathbb{B}_2} - \hat{k}_{\mathbb{B}_2} < c_5$. It follows from here and $\beta_{\mathbb{D}} = \sqrt{2}k_{\mathbb{B}_2}|_{\mathbb{D}\times\{0\}}$ that

$$b_D(z_n, w_n) + |z_n - w_n| > b_{\mathbb{D}}(z_n, w_n) - \sqrt{2}c_5$$

which, together with Lemma 1.5.4 and (1.5.18), easily implies the lower estimate in Proposition 1.5.7 if $|z_n - w_n|^2 > d_D(z_n)d_D(w_n)$.

To prove the lower estimate in Proposition 1.5.8 when $n \notin A$ and $|z_n - w_n|^2 \le d_D(z_n)d_D(w_n)$, it suffices to observe that (1.5.17) leads to $3b_D(z_n, w_n) \ge b_{\mathbb{D}}(z_n, w_n)$ and then to apply Lemma 1.5.4 and (1.5.18).

So, Proposition 1.5.8 is completely proved.

□

Recall now another comparison result between c_D and k_D (see [**Nik1**, Proposition 9]): if D is a finitely connected bounded planar domain without isolated boundary points, then

$$\lim_{\substack{w\to\partial D\\ z\neq w}} \frac{c_D(z,w)}{k_D(z,w)} = 1 \quad \text{uniformly in } z\in D. \tag{1.5.19}$$

It is worth to indicate that in [**Ven**] we spot the first tracks of the study of invariant distances in the spirit to the just quoted result.

The next proposition, which is the second and the last one result in this Section, shows that (1.5.19) remains true if we replace c_D or k_D by $b_D/\sqrt{2}$.

Proposition 1.5.18. ([**Nik-Try2**, Proposition 3]) *If D is a finitely connected bounded planar domain without isolated boundary points, then*

$$\lim_{\substack{w\to\partial D\\ z\neq w}} \frac{b_D(z,w)}{c_D(z,w)} = \lim_{\substack{w\to\partial D\\ z\neq w}} \frac{b_D(z,w)}{k_D(z,w)} = \sqrt{2} \quad \text{uniformly in } z\in D.$$

The isolated boundary points condition is essential. Indeed, if p is an isolated boundary point of a planar domain $D\neq\mathbb{C}\setminus\{p\}$ then $c_D = c_{D\cup\{p\}}$ and $b_D = b_{D\cup\{p\}}$, but $k_D(z,w)\to\infty$ as $w\to p$ and $z\in D$ is fixed.

PROOF. By the Köbe Uniformization Theorem (cf. [**Gun**, Chapter 9]), we may assume that ∂D consists of a finite number of (pairwise disjoint) $\mathcal{C}^\omega$ Jordan curves. Using Proposition 1.5.11, Corollary 1.5.12, (1.5.19) and compactness (of ∂D), it is enough to prove that

$$\lim_{\substack{z,w\to p\\ z\neq w}} \frac{b_D(z,w)}{k_D(z,w)} = \sqrt{2}$$

for any point $p\in\partial D$.

Applying the inversion, we may assume that the outer boundary of D is the unit circle and $p=1$. Then 1.5.16 and 1.5.17 imply

$$\lim_{z\to 1} \frac{\beta_{E_r}(z)}{\beta_D(z)} = 1 = \lim_{z\to 1} \frac{\kappa_{E_r}(z)}{\kappa_D(z)}.$$

The first equality shows that $\liminf_{\substack{z,w\to 1\\ z\neq w}} \frac{b_{E_r}(z,w)}{b_D(z,w)} \geq 1$.

To get that

$$\limsup_{\substack{z,w\to 1\\ z\neq w}} \frac{b_{E_r}(z,w)}{b_D(z,w)} \leq 1, \tag{1.5.20}$$

we shall follow the proof of [**Ven**, Proposition 3]. Fix an $\epsilon>0$ and choose an $r_1\in(0,r_0)$ such that

$$\beta_{E_r}(z) < (1+\epsilon)\beta_D(z), \quad z\in E_{r_1}.$$

Combining the argument in the case $n\notin A$ from the previous proof, (1.5.17), the estimates from Proposition 1.5.8, and the explicit calculations for the Bergman metric and distance on $\mathbb{D}$ (cf. [**Kra1**, Section 1.1]) we may find an $r_2 \in (0, r_1)$ such that if $z, w \in E_{r_2}$, and $\gamma : [0,1] \to D$ is a smooth curve for which $\gamma(0) = 1$, $\gamma(1) = w$, and

$$\int_0^1 \beta_D(\gamma(t); \gamma'(t))dt \leq (1+\epsilon) b_D(z,w),$$

then $\gamma([0,1]) \subset E_{r_1}$. It follows that

$$b_{E_r}(z,w) \leq \int_0^1 \beta_{E_r}(\gamma(t); \gamma'(t))dt$$

$$\leq (1+\epsilon) \int_0^1 \beta_D(\gamma(t); \gamma'(t))dt \leq (1+\epsilon)^2 b_D(z,w), \quad z, w \in E_{r_2}.$$

To obtain (1.5.20), it remains to let $\epsilon \to 0$.

So, $\lim_{\substack{z,w \to 1 \\ z \neq w}} \frac{b_{E_r}(z,w)}{b_D(z,w)} = 1$.

On the other hand, [**Nik1**, Proposition 8] gives the estimates from Proposition 1.5.8 for $2k_D$ instead of $\sqrt{2} b_D$. Then we get as above that

$$\lim_{\substack{z,w \to 1 \\ z \neq w}} \frac{\kappa_{E_r}(z,w)}{\kappa_D(z,w)} = 1.$$

Now, the equality $b_{E_r} = \sqrt{2} k_{E_r}$ completes the proof. □

1.6. Remark on the symmetrized polydisc

In this Section every time when we say boundary, closure, etc... of a domain D in $\mathbb{C}$ we mean, respectively, boundary, closure, etc... in $\hat{\mathbb{C}}$.

While we were trying to compute the Bergman metric on the symmetrized bidisc we rediscovered and generalized the characterization given by Costara in [**Cos2**]. In this Section we give different, in our opinion more elemantary, proof of this fact which is based on [**Try2**].

The Reader might find the proofs and the definitions from this Section in [**Mar**] and [**Pól-Sze**] as well.

By a *circular domain* we mean a domain in $\mathbb{C}$ which is closed interior or exterior of a disc or a halfplane and by *circle* we mean the boundary (in $\hat{\mathbb{C}}$) of any circular domain.

Let $z_1, \ldots, z_n$ be arbitrary points in $\hat{\mathbb{C}}$ (not necessarily finite), $z \neq z_j$, $j = 1, \ldots, n$ and let $m_1, \ldots, m_n$ be non-negative numbers (masses) of total sum (mass) 1 which are placed at points $z_1, \ldots, z_n$, respectively. Choose any linear fractional transformation of complex plane L which sends z to ∞ (that is L is of the form $\frac{a \cdot + b}{c \cdot + b}$). *By center of gravity* ζ of such a mass-distribution with respect to z we understand a point $\zeta := \zeta_z$ such that $L(\zeta)$ is an ordinary center of gravity of $L(z_1), \ldots, L(z_n)$ with masses $m_1, \ldots, m_n$. Note that point ζ does not depend on the choice of L. It is worth mentioning that ordinary center of gravity is a case when $z = \infty$.

Consider all possible mass disstributions with total mass 1 over the fixed points $z_1, \ldots, z_n$ and the point of reference z distinct from all z_ν. Set C_z consisting of the centers of gravity ζ_z of all mass distributions of this kind is called a *circular-arc polygon.* Geometrical interpretation of that definition is contained in the following

Lemma 1.6.1 ([**Pól-Sze**]). *For any two points $w_1, w_2 \in C_z$* ***arc of circle through*** *w_1, w_2, z* ***with end-points*** *w_1, w_2* ***that does not contain*** *z, is contained in C_z.*

A set with the property described in Lemma 1.6.1 is called *circularly(-arc) convex with respect to z.*

Someone might easily check that the set C_z is the smallest circularly-convex domain with respect to z that contains the points $z_1, \dots, z_n$. When $z = \infty$, C_z is just a convex hull $\operatorname{conv}(z_1, \dots, z_n)$, and circular-convexity is reduced to convexity in an ordinary sense.

So, we get

Lemma 1.6.2. *If the points $z_1, \dots, z_n$ lie in a circular domain C but z lies in the complement circular domain to C then $C_z \subset C$.*

From now on, by a center of gravity we mean the center with special mass distribution $m_1 = \ldots = m_n = \frac{1}{n}$.

Lemma 1.6.3 ([**Pól-Sze**]). *Let ζ_z be the center of gravity of $z_1, \dots, z_n$ with respect to z. Every circle through z and ζ_z either separates the points $z_1, \dots, z_n$ or all the points lie on the circle. Moreover, if $z_1, \dots, z_n$ belong to a circular domain C, then points z, ζ_z cannot both lie outside C.*

Let f be any polynomial of degree n:

$$f(z) = C(n,0)A_0^{(0)} + C(n,1)A_1^{(0)}z + \ldots + C(n,n)A_n^{(0)}z^n, \tag{1.6.1}$$

where $C(n,k)$ is the binomial coefficient (it is possible that $A_n^{(0)} = \ldots = A_{n-k+1}^{(0)} = 0$, $A_{n-k}^{(0)} \neq 0$, and then ∞ is interpreted as a k fold zero of f). Point ζ_z is a *center of gravity of a polynomial with respect to z* if it is the center of gravity of its zeros with respect to z.

Take any point ζ. *Polar derivative $A_\zeta f$* of f with respect to z is defined by the equality

$$(\zeta - z)f'(z) + nf(z) = A_\zeta f(z) \quad \text{if } \zeta \in \mathbb{C}, \tag{1.6.2}$$

or just $f'(z)$ if $\zeta = \infty$. Notice that $\deg A_\zeta f < \deg f$ if $A_n^{(0)} \neq 0$. Let points $\zeta_1, \dots, \zeta_{k+1}$ be given, $(k+1)$*th polar derivative* f is defined as

$$A_{\zeta_1} \ldots A_{\zeta_{k+1}} f = A_{\zeta_{k+1}}(A_{\zeta_1} \ldots A_{\zeta_k} f).$$

In fact, the order of points $\zeta_1, \dots, \zeta_{k+1}$ is not important, that is the operations A_{ζ_1} and A_{ζ_2} are commutative. Actually, using induction one might show

$$A_{\zeta_1} \ldots A_{\zeta_k} f(z) = C(n,k)k! \sum_{j=0}^{n-k} C(n-k,j)A_j^{(k)} z^j, \tag{1.6.3}$$

where

$$A_j^{(k)} = \sum_{l=0}^{k} s_l^{(k)}(\zeta_1, \dots, \zeta_k) A_{j+l}^{(0)}. \tag{1.6.4}$$

If points $\zeta_1 = \ldots = \zeta_m = \infty$ then $s_l^{(k)}(\zeta_1, \ldots, \zeta_k) := 0$ for $l < m$ and $s_l^{(k)}(\zeta_1, \ldots, \zeta_k) := s_{l-m}^{(k-m)}(\zeta_{m+1}, \ldots, \zeta_k)$, and the last one are elementary symmetric polynomials (see Section 1.1).[(3)]

Recall the well known Gauss-Lucas Theorem, which states that every convex set which contains all zeros of a given polynomial f also contains f''s critical points (i.e. zeros of f'). For polar derivative similar result holds. Namely

Theorem 1.6.4. (Laguerre cf. [**Pól-Sze**]) *If all the zeros of a polynomial f of degree n lie in a circular domain C and if Z is any zero of $A_\zeta f$ then not both points Z, ζ may lie outside C. Furthermore, if $f(Z) \neq 0$, then any circle through Z and ζ either passes through all the zeros of f or separates these zeros.*

We say that polynomial g is *apolar* to polynomial f (both of them are of degree n) if nth polar derivative of f counted with respect to the zeros of g vanishes. Notice that g is apolar to f if and only if f is apolar to g, and we express this fact saying that f and g are *apolar*.

Lemma 1.6.5. [**Pól-Sze**, pg. 60] *Let f be a polynomial given as in (1.6.1). Assume that f is apolar to g, where*

$$g(z) = \sum_{j=0}^{n} C(n,j) B_j^{(0)} z^j.$$

Then every two circularly-arc polygons that are circularly convex with respect to the same point and that contain all the zeros of f and g respectively, also have at least one common point.

Let $(s_1, \ldots, s_n) \in \mathbb{C}^n$. Define $P(z) = z^n - s_1 z^{n-1} + \ldots + (-1)^n s_n$. Then the Theorem 3.1 from [**Cos2**] can be generalized as follows

Proposition 1.6.6. *$P^{-1}(0) \subseteq \mathbb{D}(z_0, r)$ if and only if*

$$\sup_{z:|z-z_0|\geq r} \left| \frac{A_{z_0} P(z)}{P'(z)} \right| =: f(z) < r. \tag{1.6.5}$$

Let $P(z) = \sum_{k=0}^{n} C(n,k) a_j z^j$ then

$$\frac{A_{z_0} P(z)}{P'(z)} = \frac{\sum_{k=0}^{n-1} C(n-1,k) a_k z^k}{\sum_{k=0}^{n-1} C(n-1,k) a_{k+1} z^k}. \tag{1.6.6}$$

In [**Cos2**] only the case when $z_0 = 0$ and $r = 1$ is discussed.

Proof. Let points $z_1, \ldots, z_n$ be all zeros of P and fix any z outside or on $C := \partial\mathbb{D}(z_0, r)$. Then, in view of Lemma 1.6.3, it is enough to notice that

$$\left| \frac{A_{z_0} P(z)}{P'(z)} \right| = |\zeta_z - z_0|, \tag{1.6.7}$$

where ζ_z is the center of gravity P with respect to z. □

(3) Here to avoid some unwanted reductions we must all the time control the number of points $z_1, \ldots, z_n$. Because of this we decided to index the elementary symmetric functions by two numbers.

Using the same argument as above we get:

Corollary 1.6.7. *Let $z_0 \in \mathbb{C},\ r > 0$. Then $P^{-1}(0) \subseteq \overline{\mathbb{D}}(z_0, r)$ if and only if*

$$\sup_{z:|z-z_0|\geq r} \left| \frac{A_{z_0}P(z)}{P'(z)} \right| \leq r.$$

Corollary 1.6.8. *Let $z_0 \in \mathbb{C},\ r > 0$. $P^{-1}(0) \subseteq \partial\mathbb{D}(z_0, r)$ if and only if $\zeta_z \epsilon\, \partial\, \mathbb{D}(z_0, r)$ and $(P')^{-1}(0) \subseteq \overline{\mathbb{D}}(z_0, r)$ for all $z \,\epsilon\, \partial\mathbb{D}(z_0, r)$.*

Proof. Assume that points $P^{-1}(0)$ lie on a circle $|z - z_0| = r$. Then ζ_z also lies on this circle. Conversly, from Corollary 1.6.7 we obtain $P^{-1}(0) \subseteq \overline{\mathbb{D}}(z_0, r)$. If $P(\widetilde{z}) = 0$, then $\widetilde{z}$ must lies on the boundary of that disc. Indeed, because $|\zeta_z - z_0| < r$ for any $z \,\epsilon\, \partial\overline{\mathbb{D}} \setminus P^{-1}(0)$. □

If we put in Proposition 1.6.5 $z_0 = 0$, $r = 1$ then we obtain a characterization of $\mathbb{G}_n$ over the unit disc. To get a characterization over $\mathbb{G}_{n-1}$ we use $(n-1)$th polar derivative.

Proposition 1.6.9. *Let P be a polynomial of degree n with complex coefficients. Then $P^{-1}(0) \subseteq \mathbb{D}(z_0, r)$ if and only if there exists $0 < s < r$ such that the only zero of $A_{\zeta_1} \dots A_{\zeta_{n-1}}P$ is in $\mathbb{D}(z_0, s)$ for all $\zeta_1, \dots, \zeta_{n-1} \notin \mathbb{D}(z_0, r)$.*

Proof. The only zero of $A_{\zeta_1} \dots A_{\zeta_{n-1}}P$ is

$$-\frac{A_{z_0}A_{\zeta_1} \dots A_{\zeta_{n-1}}P}{A_\infty A_{\zeta_1} \dots A_{\zeta_{n-1}}P} =: g(\zeta_1, \dots, \zeta_{n-1}).$$

After some straightforward calculation we check that $g(\zeta, \dots, \zeta) = f(\zeta)$, where f is as in Proposition 1.6.6. Applying Lemma 1.6.3 $(k-1)$ times gives 'only if'. It remains to show the sufficiency of the above condition. For this part, notice that (1.6.3) and (1.6.4) imply $A_{\widetilde{z}}^{n-1}P(\widetilde{z}) = P(\widetilde{z})$ for any $\widetilde{z}$. □

Lemma 1.6.3 gives the following generalization of Propositions 1.6.6 and 1.6.7. It extends the characterization from [**Cos2**]. We skip the proof because it is similar to the proofs given above.

Proposition 1.6.10. *Let f be any polynomial of degree n with coefficient at z^n equal to 1. The following assertions are equivalent:*

(1) $P^{-1}(0) \subset \mathbb{D}(z_0, r)$;

(2)

$$\sup_{z\notin\mathbb{D}(z_0,r)} \left| \frac{A_{z_0}A_{\zeta_1} \dots A_{\zeta_{k-1}}f(z)}{A_\infty A_{\zeta_1} \dots A_{\zeta_{k-1}}f(z)} \right| < r$$

for any positive integer number $1 \leqslant k \leqslant n-1$ and any choice of the points $\zeta_1, \dots, \zeta_{k-1} \notin \mathbb{D}(z_0, r)$;

(3) (2) holds for $k = 1$;

(4) (2) holds for $k = n-1$;

(5) (2) holds for $k = n-1$ and $\zeta_1 = \ldots = \zeta_{n-1} \notin \mathbb{D}(z_0, r)$;

(6) (2) holds for some $1 \leqslant k \leqslant n-1$;

(7) (2) holds for some $1 \leqslant k \leqslant n-1,\ \zeta_1 = \ldots = \zeta_k \notin \mathbb{D}(z_0, r)$.

CHAPTER 2

On other holomorphically invariant distances

In Chapter 1 we pondered over the Bergman invariants. Now we concentrate on the geometry of another holomorphically invariant distance which is of great interest in complex analysis, i.e. the Kobayashi distance.

2.1. Geometry of the Kobayashi distance

This Section uses the power of the geometry of 'convex' sets. First, we introduce some notions of convexity that we work with hereafter. Then we provide the basic properties of just introduced objects. After that we state and prove Theorem 2.1.3 which is the main result of this Section.

Fix a domain D in $\mathbb{C}^n$.

D is said to be:

• $\mathbb{C}$-*convex* if any non-empty intersection with a complex line is a simply connected domain.

• *linearly (weakly linearly convex) convex* if for any $a \in \mathbb{C}^n \setminus D$ ($a \in \partial D$) there exists a complex hyperplane passing through a which does not intersect D.

Observation 2.1.1. *(1)*

$$\text{convexity} \;\Rightarrow\; \mathbb{C}-\text{convexity} \Rightarrow\; \text{linear convexity} \Rightarrow\; \text{weak linear convexity}.$$

Only the second implication might cause any difficulties (for the proof cf. [**A-P-S**, Theorem 2.3.9(ii)]). Moreover, in the case of C^1-smooth bounded domains the last three notions coincide (see [**A-P-S**, Corollary 2.5.6]).

(2) Projections preserve $\mathbb{C}$-convexity (cf. [**A-P-S**, Theorem 2.3.6]).

(3) Suppose that the weakly linearly convex domain $D \subset \mathbb{C}^n$ contains the n unit discs lying in the coordinate lines. Then D contains the convex hull of these discs $E = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j| < 1\}$ (cf. [**Zna-Zna**]).

To avoid unwanted reductions assume that D is *proper*, i.e. D contains no complex affine line.

A good tool in the studies of the geometry of $\mathbb{C}$-convex sets is the so-called *minimal basis*. However, we indicate that the construction is correct for every domain in $\mathbb{C}^n$. To introduce it let us fix a point $q \in D$. Choose $q^1 \in \partial D$ so that $\tau_1(q) := |q^1 - q| = d_D(q)$. Put $H_1 = q + \operatorname{span}(q^1 - q)^\perp$ and $D_1 = D \cap H_1$. Let $q^2 \in \partial D_1$ be so that $\tau_2(q) := |q^2 - q| = d_{D_1}(q)$. Put $H_2 = q + \operatorname{span}(q^1 - q, q^2 - q)^\perp$, $D_2 = D \cap H_2$ and so on. Thus we get the vectors $e_j = \frac{q^j - q}{|q^j - q|}$, $1 \le j \le n$ which make up an orthonormal basis of $\mathbb{C}^n$. The basis we call *minimal* for D at q. Furthermore, we get positive numbers $(\tau_1(q) \le \tau_2(q) \le \cdots \le \tau_n(q)) =: \tau(q)$. The basis and the numbers are not uniquely determined. However, after rotation we may replace $e_1, e_2, \dots, e_n$ by the standard basis of $\mathbb{C}^n$.

The construction of minimal basis was intitiated in [**McN**]. However, there was some flaw in McNeal's reasoning. Later, Nikolov with Pflug corrected the construction (see [**Nik-Pfl**]).

We recall only one result concerning the minimal basis which we apply in the next Section.

Proposition 2.1.2. ([**N-P-Z2**]) *There exists a constant $c_n \geq 1$ (depending only on n) such that for each $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, not containing a complex line, we have*

$$c_n^{-1} \leq F_D(q;X)\Big(\sum_{j=1}^{n} \frac{|\langle X, e_j(q)\rangle|}{\tau_j(q)}\Big)^{-1} \leq c_n,$$

for $q \in D$, $X \in \mathbb{C}^n$, $X \neq 0$, where $F_D = \gamma_D$, $F_D = \beta_D$, or $F_D = \kappa_D$.

The main result of the present Section is

Theorem 2.1.3. ([**Nik-Try1**]) *Assume that D contains no complex line. Fix $q \in D$. Assume that the standard basis of $\mathbb{C}^n$ is minimal for D at q. Let $r > 0$.*

(i) If D is weakly linearly convex then the following implications hold for $z \in \mathbb{C}^n$

$$\max_{1\leq j\leq n} \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r}-1}{n(e^{2r}+1)} \Rightarrow \sum_{j=1}^{n} \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r}-1}{e^{2r}+1}$$
$$\Rightarrow z \in D \text{ and } l_D(q,z) < r.$$

(ii) If D is convex and $z \in D$ then the property $c_D(q,z) < r$ implies

$$\max_{1\leq j\leq n} \frac{|z_j - q_j|}{\tau_j(q)} < e^{2r} - 1.$$

(iii) If D is $\mathbb{C}$-convex and $z \in D$ then the property $c_D(q,z) < r$ implies

$$\max_{1\leq j\leq n} \frac{|z_j - q_j|}{\tau_j(q)} < e^{4r} - 1.$$

Theorem 2.1.3 gives information about the sizes of Kobayashi balls. More precisely, if

- D is convex then the following inclusions hold

$$\mathbb{D}^n\Big(q, \frac{1}{n}\frac{e^{2r}-1}{e^{2r}+1}\tau(q)\Big) \subset \mathbb{B}_{l_D}(q,r) \subset \mathbb{B}_{k_D}(q,r) \subset \mathbb{B}_{c_D}(q,r) \subset \mathbb{D}^n\Big(q, (e^{2r}-1)\tau(q)\Big),$$

By Lempert Theorem the second and the third inclusion we may replace by equality (the Reader is invited to consult [**Lem1**], [**Lem2**] and [**Jar-Pfl2**, Chapter 8]). However, in the thesis we tried to do without this strong and powerful Theorem.

- D is $\mathbb{C}$-convex then

$$\mathbb{D}^n\Big(q, \frac{1}{n}\frac{e^{2r}-1}{e^{2r}+1}\tau(q)\Big) \subset \mathbb{B}_{l_D}(q,r) \subset \mathbb{B}_{k_D}(q,r) \subset \mathbb{B}_{c_D}(q,r) \subset \mathbb{D}^n\Big(q, (e^{4r}-1)\tau(q)\Big).$$

Theorem 2.1.3 is a consequence of the following lemma which is interesting in its own.

Lemma 2.1.4. ([**Nik-Try1**]) *(i) Let D be a proper convex domain in $\mathbb{C}^n$. Then*

$$c_D(z,w) \geq \frac{1}{2}\log\frac{d_D(z)}{d_D(w)}.$$

Moreover, if $n=1$, then

$$c_D(z,w) \geq \frac{1}{2}\log\left(1+\frac{|z-w|}{d_D(w)}\right).$$

(ii) Let D be a proper $\mathbb{C}$-convex domain in $\mathbb{C}^n$. Then

$$c_D(z,w) \geq \frac{1}{4}\log\frac{d_D(z)}{d_D(w)}.$$

Moreover, if $n=1$, then

$$c_D(z,w) \geq \frac{1}{4}\log\left(1+\frac{|z-w|}{d_D(w)}\right).$$

The constants $1/2$ and $1/4$ are sharp as the examples $D=\mathbb{D}$ and $D=\mathbb{C}_*\setminus\mathbb{R}_{>0}$ show. Note that in the $\mathbb{C}$-convex case the weaker estimate

$$c_D(z,w) \geq \frac{1}{4}\log\frac{d_D(z)}{4d_D(w)}$$

is contained in [**Nik1**, Proposition 2]

Lemma 2.1.4 relies on the following

Lemma 2.1.5. ([**Jar-Pfl2**, Lemma 4.3.3) (c, d)]) *Let G be a domain in $\mathbb{C}^n$ and let d_G be a pseudodistance on G satisfying the following property*

if $\mathbb{B}(a,r)\subset G$ then there exists a constant $M>0$ such that
$$d_G(z,w)\leq M|z-w| \text{ for } z,\ w\in\mathbb{B}(a,r).$$

Define a metric δ_G by a formula

$$\delta_G(a;X) = \limsup_{\lambda\to 0,\, z\to a}\frac{1}{|\lambda|}d_G(z,z+\lambda X),\quad a\in G,\ X\in\mathbb{C}^n.$$

Then the δ_G is an upper semicontinous pseudometric and

$$\begin{aligned}&d_G(z,w)\\ &\leq \inf\left\{\int_{[0,1]}\delta_G(\alpha(t);\alpha'(t))dt : \alpha:[0,1]\to G \text{ piecewise } \mathcal{C}^1,\ \alpha(0)=z,\ \alpha(1)=w\right\}\end{aligned}$$

for every z, $w\in G$.

Proof of Lemma 2.1.4. After translation and rotation, we may assume that $0\in\partial D$ and $w=(d_D(w),0,\dots,0)$.

(i) We have that $D \subset \Pi^+ = \{\zeta \in \mathbb{C}^n : \operatorname{Re} \zeta_1 > 0\}$ and hence by remarks before Lemma 1.5.4

$$c_D(z,w) \geq c_{\Pi^+}(z,w) = \tanh^{-1}\left|\frac{z_1 - w_1}{z_1 + \overline{w}_1}\right|$$
$$\geq \tanh^{-1}\frac{|z_1 - w_1|}{|z_1 - w_1| + 2d_D(w)} = \frac{1}{2}\log\left(1 + \frac{|z_1 - w_1|}{d_D(w)}\right).$$

(ii) It follows by weak linear convexity that $D \cap \{\zeta_1 \in \mathbb{C}^n : \zeta_1 = 0\} = \varnothing$. Denote by D_1 the projection of D onto the ζ_1-plane. The Köbe Quarter Theorem implies that

$$\gamma_{D_1}(\zeta_1; e_1) \geq \frac{1}{4d_{D_1}(\zeta_1)} \geq \frac{1}{4|\zeta_1|}.$$

Since D_1 is a simply connected domain then

$$c_D(z,w) \geq c_{D_1}(z_1,w_1) = \inf_s \int_0^1 \gamma_{D_1}(s(t); s'(t)dt \geq \frac{1}{4}\inf_s \int_0^1 \left|\frac{s'(t)}{s(t)}\right| dt,$$

where the infimum is taken over all $\mathcal{C}^1$ curves $s : [0,1] \to D_1$ with $s(0) = z_1$ and $s(1) = w_1$.

Set now

$$d(\zeta_1, \eta_1) = \log\max(1 + |1 - \zeta_1/\eta_1|, 1 + |1 - \eta_1/\zeta_1|).$$

It is easy to check that d is a distance on $\mathbb{C}_*$ with "derivative"[(1)]

$$\lim_{\lambda \to 0,\, \lambda \neq 0} \frac{d(\zeta_1, \zeta_1 + \lambda)}{|\lambda|} = \frac{1}{|\zeta_1|}.$$

Then by Lemma 2.1.5

$$\inf_s \int_0^1 \left|\frac{s'(t)}{s(t)}\right| dt \geq d(z_1, w_1)$$

and hence

$$c_D(z,w) \geq \frac{1}{4}d(z_1,w_1) \geq \frac{1}{4}\log\left(1 + \frac{|z_1 - w_1|}{d_D(w)}\right).$$

□

Proof of Theorem 2.1.3. (i) Since D contains the discs $\mathbb{D}(q_1, \tau_1(q)), \dots,$ $\mathbb{D}(q_n, \tau_n(q))$ (lying in the respective coordinate complex planes), it contains their convex hull

$$C = \{\zeta \in \mathbb{C}^n : h(\zeta) = \sum_{j=1}^n \frac{|\zeta_j - q_j|}{\tau_j(q)} < 1\}.$$

Then

$$l_D(q,z) \leq l_C(q,z) = \tanh^{-1} h(z)$$

(this is the consequence of the formula for the Lempert function for balanced domains, cf. [**Jar-Pfl2**, Proposition 3.1.10]) which implies (i).

---

(1) Let $a, b, c \in \mathbb{C}_*$ and $d_1 = 1 - a/b$, $d_2 = 1 - b/c$, $d_3 = 1 - a/c$. We may assume that $d(a,c) = \log(1 + |d_3|)$. Then

$$d(a,b) + d(b,c) \geq \log(1 + |d_1|) + \log(1 + |d_2|)$$
$$= \log(1 + |d_1| + |d_2| + |d_3 - d_2 - d_1|) \geq \log(1 + |d_3|) = d(a,c).$$

Before proving (ii) and (iii) note that by ($\mathbb{C}$-)convexity and the construction of the minimal basis there exists a complex hyperplane $q^{j+1}+W_j$ through q^{j+1} that is disjoint from D, $j=0,\dots,n-1$. It is not difficult to see that W_j is given by the equation

$$\alpha_{j,1}\zeta_1+\cdots+\alpha_{j,j}\zeta_j+\zeta_{j+1}=0.$$

Let $\Lambda:\mathbb{C}^n\to\mathbb{C}^n$ be the linear mapping with matrix whose rows are given by the vectors $(\alpha_{j,1},\dots,\alpha_{j,j},1,0,\dots,0)$. Set $\Lambda_q(\zeta)=q+\Lambda(\zeta-q)$. Note that $G=\Lambda_q(D)$ is a ($\mathbb{C}$-)convex domain. Denote by G_j the projection of G onto j-th coordinate plane. Then $G\subset G'=G_1\times\cdots\times G_n$ and the product formula for the Carathéodory distance implies that

$$c_D(q,z)\geq c_{G'}(q,\Lambda_q(z))=\max_{1\leq j\leq n}c_{G_j}(q_j,z_j). \tag{2.1.1}$$

Observe also that $d_{G_j}(q_j)=\tau_j(q)$.

(ii) If D is a convex domain then G_j is a convex domain. Hence, by Lemma 2.1.5

$$c_{G_j}(q_j,z_j)\geq\frac{1}{2}\log\left(1+\frac{|z_j-q_j|}{\tau_j(q)}\right)$$

and (ii) follows from here and (2.2.3).

(iii) If D is a $\mathbb{C}$-convex domain, then G_j is a simply connected domain. Hence, by Lemma 2.1.5,

$$c_{G_j}(q_j,z_j)\geq\frac{1}{4}\log\left(1+\frac{|z_j-q_j|}{\tau_j(q)}\right)$$

and (iii) follows from here and (2.2.3). □

Theorem 2.1.3 has a local version which we state below. Before we do it some explanation is required. By any of the notion of convexity (introduced at the begining of the Section) near some boundary point a we mean that there exists a neighborhood U of a such that $D\cap U$ is an open set with the respective global convexity.

Theorem 2.1.6. ([**Nik-Try1**]) *Let D be a domain in $\mathbb{C}^n$ whose boundary contains no affine disc through $a\in\partial D$. Assume that the standard basis of $\mathbb{C}^n$ is minimal for D at $q\in D$. Let $r>r'>0$.*

(i) If D is weakly linearly convex near a then the following implications hold

$$\max\nolimits_{1\leq j\leq n}\frac{|z_j-q_j|}{\tau_j(q)}<\frac{e^{2r}-1}{n(e^{2r}+1)}\Rightarrow\sum_{j=1}^{n}\frac{|z_j-q_j|}{\tau_j(q)}<\frac{e^{2r}-1}{e^{2r}+1}$$

$$\Rightarrow z\in D \text{ and } l_D(q,z)<r$$

for q sufficiently close to a.

(ii) If D is convex near a then the inequality $k_D(q,z)<r'$ implies $\max_{1\leq j\leq n}\frac{|z_j-q_j|}{\tau_j(q)}<e^{2r}-1$ for q sufficiently close to a.

(iii) If D is $\mathbb{C}$-convex near a and bounded then the inequality $k_D(q,z)<r'$ implies $\max_{1\leq j\leq n}\frac{|z_j-q_j|}{\tau_j(q)}<e^{4r}-1$ for q sufficiently close to a.

Proof of Theorem 2.1.6 is merely an application of Theorem 2.1.3. Because of this we skip it.

2.2. Gromov hyperbolicity

There are a number of equivalent ways of formulating the hyperbolicity condition. One of possible condition is the following.

Definition 2.2.1. Let (X, d) be a metric space and let $\delta > 0$. A *geodesic triangle* is the union of three pairwise intersecting curves such that the length of its every side is equal to the distance between the end points. (If $\alpha : [0, 1] \to X$ is a curve, then the number

$$L_d(\alpha) := \sup \Big\{ \sum_{j=1}^{N} d(\alpha(t_{j-1}), \alpha(t_j)) : N \in \mathbb{N},\, 0 = t_0 < t_1 < \ldots < t_N = 1 \Big\}$$

is called *d-length of* α.)

A metric space (X, d) is called *geodesic* if for every two points $a, b \in X$ there exists a curve with length equal to $d(a, b)$.
A geodesic triangle in a metric space is said to be $\delta-$slim if each of its sides is contained in the $\delta-$neighbourhood of the union of the other two sides. A geodesic space X is said to be $\delta-$hyperbolic if every triangle in X is $\delta-$slim. If X is $\delta-$hyperbolic for some $\delta > 0$, one often says simply that X is hyperbolic. See Figure 1.

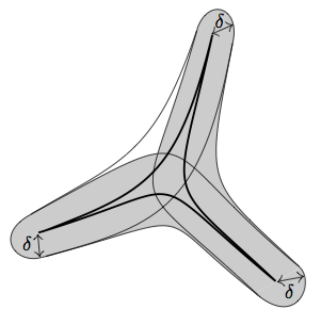

FIGURE 1. $\delta-$slim triangle

However, hereafter we prefer working with less restrictive

Definition 2.2.2. Let (D, d) be a metric space. Given points $x, y, z \in D$, the *Gromov product* is

$$(x, y)_z := d(x, z) + d(z, y) - d(x, y).$$

Let

$$S(p, q, x, w) := \min\big((p, x)_w, (x, q)_w\big) - (p, q)_w.$$

D is *Gromov hyperbolic* with respect to d if and only if

$$\sup_{p,q,x,w \in D} S(p, q, x, w) < \infty. \tag{2.2.1}$$

If $S(p, q, x, w) \le 2\delta$, we say that D is δ-hyperbolic (possibly with $\delta = 0$).

At first glance, there is a big difference between both definitions. The first one requires the knowledge about all geodesics, and is rather useful for indicating the non-hyperbolicity of a space than the hyperbolicity. The second one is very general and might be applied to any metric space, not necessarily coming from the (smooth) metric (see Section 2.3 and Proposition 2.2.11). But both of them are very hard to check.

If (X, d) is an *intristic metric space*, what means that the distance between two points is always equal to the infimum of the lengths of all arcs joining them, then both Definitions are equivalent (cf. [**Bow**]). In particular, it is the case when we work with the Kobayashi distance.

Example 2.2.3. The prototype of a Gromov hyperbolic space is a simply connected complete Riemannian manifold with curvature bounded above by a negative constant (cf. [**Gro1**, pg. 76])

$$K \leq -\delta^2 < 0. \tag{2.2.2}$$

If (2.2.2) is satisfied then the space is $(C\delta)-$hyperbolic, where $C \in (1, 10)$ (cf. [**Gro1**, 1.5(1)]).

Example 2.2.4. Let us consider $\mathbb{R}^n$ with the standard Euclidean distance $|\ |$. Geodesic lines in $(\mathbb{R}^n, |\ |)$ are the segments. Since $[a, b] \subset [a, c] \cup [c, b]$ for every three arbitrarily chosen points a, b, $c \in \mathbb{R}$, we have that $\mathbb{R}$ is $0-$hyperbolic. In higher dimensions the situation is totally different. Actually, ponder an equilateral triangle ABC with sides of length k. Consider the orthogonal projection of A onto BC and call it D. Then

$$\operatorname{dist}(D, [A, B] \cup [A, C]) \to \infty \quad \text{as} \quad k \to \infty.$$

Recall that $\operatorname{length}(\alpha) = \int_0^1 |\alpha'(t)| dt$ for every $\mathcal{C}^1$-piecewise curve α, thus the curvature of $|\ |$ is 0. This shows that in (2.2.2) the assumption about strict negative curvature can not be weakened.

Example 2.2.5. $(\mathbb{D}, k_{\mathbb{D}})$ is $(\sqrt{2} + 1)$-hyperbolic (cf. [**Lon**]).

An immediate consequence of Definition 2.2.2 is

Corollary 2.2.6. *Let D be a non-empty bounded open set in $\mathbb{C}^n$ and let d be a distance on D. If $d : (D, |\,|) \to (\mathbb{R}, |\,|)$ is continuous then it is sufficient to check the condition* (2.2.1) *near the boundary $\partial(D \times D)$.*

Observe that the Kobayashi distance satisfies the assumptions of Corollar 2.2.6 (see [**Jar-Pfl2**, Proposition 3.1.9]).

Example 2.2.7. Fix $r > 1$ and put $\mathbb{A} = \{z \in \mathbb{C} : 1/r < |z| < r\}$. Recall that we have the localization property of the Kobayashi metric on $\mathbb{A}$ (Theorem 1.5.14). Since both definitions are invariant under biholomorphic mappings, near the outer boundary $\{|z| = r\}$ the metric on $\mathbb{A}$ behaves like on the disc, and (after the inversion $z \to \frac{1}{z}$) near the inner boundary, too.

Theorem 2.2.8. (cf. [**Väi**, Theorem 3.18, Theorem 3.20]) *Let (X, d_X) and (Y, d_Y) be intrinsic spaces. Let $f : X \to Y$ be a (c, M)-quasi isometry (i.e. $\frac{1}{c}\, d_X(x, y) - M \leq d_Y\big(f(x), f(y)\big) \leq c\, d_X(x, y) + M,\ x,\, y \in X$). If Y is hyperbolic then X is as well.*

Recall that the Carathéodory, the Kobayashi and the Bergman distances on convex domains, or more generally on $\mathbb{C}$-convex domains containing no complex lines, are bilipschitz equivalent [**N-P-Z2**, Theorem 12]. Recall that two metrics s_D and r_D on a set D are *bilipschitz equivalent* if the identity map $\mathrm{id} : (D, s_D) \to (D, r_D)$ is a $(c, 0)$-quasi isometry for some $c \geq 1$. Consequently, it is enough to derive the hyperbolicity on $\mathbb{C}$-convex domains for one invariant distance.

The first work concerning Gromov hyperbolicity on domains endowed with Kobayashi distance was given by Balogh and Bonk [**Bal-Bon**] who gave both positive and negative examples. Among other results, they proved that the Cartesian product of strictly pseudoconvex domains is not Gromov hyperbolic. It is a special case of a general situation mentioned in many places but without proof (cf. [**Gau-Ses**]).

Proposition 2.2.9. *Assume that (X_1, d_1) is an intrinsic metric space with d_1 unbounded and assume that (X_2, d_2) is a metric space with unbounded d_2. Let $d = \max\{d_1, d_2\}$. Then $(X_1 \times X_2, d)$ is not Gromov hyperbolic.*

Proof. Assume that d is $\frac{\delta}{2}$-Gromov hyperbolic. Put $k = 3 + \delta$. Then there are points $y_1, y_2 \in X_2$ such that $d_2(y_1, y_2) = 2s \geq 2k$. Choose points $x_1, x_2^* \in X_1$ with $d_1(x_1, x_2^*) \geq 2s$. By the path property of X_1, there is a curve $\gamma : [0, 1] \to X_1$ joining the points x_1 and x_2^* such that $L_{d_1}(\gamma) < d_1(x_1, x_2^*) + 1$. Note that the function $t \to d_1(x_1, \gamma(t))$ is continuous. Hence there is the smallest t_0 such that $d_1(x_1, \gamma(t_0)) = 2s$. Set $x_2 := \gamma(t_0)$.

Now $L_{d_1}(\gamma|_{[0,t_0]}) \geq d_1(x_1, x_2) = 2s$, and

$$L_{d_1}(\gamma|_{[0,t_0]}) = L_{d_1}(\gamma) - L_{d_1}(\gamma|_{[t_0,1]}) \leq d_1(x_1, x_2^*) + 1 - d_1(x_2, x_2^*) \leq d_1(x_1, x_2) + 1.$$

Let t_1 be the smallest number in $[0, t_0]$ such that $d_1(x_1, \gamma(t_1)) = s$. Set $x_3 := \gamma(t_1)$. Then

$$d_1(x_2, x_3) \geq d_1(x_1, x_2) - d_1(x_1, x_3) = s, \text{ and}$$

$$d_1(x_2, x_3) = L_{d_1}(\gamma|_{[0,t_1]}) = L_{d_1}(\gamma|_{[0,t_0]}) - L_{d_1}(\gamma|_{[t_1,t_0]}) \leq 2s + 1 - d_1(x_1, x_2) = s + 1.$$

Hence, $s = d_1(x_1, x_3) \leq d_1(x_3, x_2) < s + 1$.

Now define the following points in $X_1 \times X_2$: $x := (x_1, y_1)$, $y := (x_2, y_1)$, $w := (x_3, y_1)$, and $z := (x_3, y_2)$. Then $d(z, w) = d(z, x) = d(z, y) = 2s$ and $(x, y)_w \leq 1$, $(x, z)_w = d(x, w) = s$, $(y, z)_w = d(y, w) \geq s - 1$. By the assumption of $\frac{\delta}{2}$-hyperbolicity we reach the following inequality

$$1 \geq (x, y)_w \geq \min\{(y, z)_w, (x, z)_w\} - \delta \geq s - 1 - \delta \geq 2;$$

a contradiction. □

The next proposition is more general than the previous one. However, its proof uses Proposition 2.2.9.

Proposition 2.2.10. *Let (X_1, d_1) and (X_2, d_2) be metric spaces, such that at least one of them is intrinsic. Then $(X_1 \times X_2, d)$ is Gromov hyperbolic if and only if one of the factors is Gromov hyperbolic and the metric of the second one is bounded.*

PROOF. Let first X_1 be 2δ-hyperbolic and $d_2 \leq 2c$. Since $d \leq d_1 + 2c$, it follows that

$$(x_1, y_1)_{w_1} - 2c \leq (x, y)_w \leq (x_1, y_1)_{w_1} + 4c$$

and then (X, d) is $(\delta + 3c)$-hyperbolic.

Assume now that (X, d) is δ-hyperbolic. Following the proof of Proposition 2.2.9, we deduce that one of the distances is bounded, say $d_2 \leq 2c$. Then we get as above that (X_1, d_1) is $(\delta + 3c)$-hyperbolic. $\square$

From this moment until the end of the Section we assume that $D \subset \mathbb{C}^n,\ d = k_D$.

As an immediate consequence of Proposition 2.2.10 we obtain that the polydisc is not hyperbolic. Moreover, even its "symmetrized" counterpart is not.

Proposition 2.2.11. *$\mathbb{G}_n$ is not Gromov hyperbolic with respect to the Carathéodory and the Kobayashi distances for $n \geq 2$. Moreover, $\mathbb{G}_2$ is not Gromov hyperbolic with respect to the Bergman distance.*

PROOF. Fix $a \in \mathbb{D}$. Put $p_a = \pi(a, \dots, a),\ q_a = \pi(a, \dots, a, -a),\ m_a = \pi(a, \dots, a, 0)$. The holomorphic contractibility and the product property gives that

$$\begin{aligned} k_{\mathbb{D}}(\pi_n(z), \pi_n(w)) &\leq k_{\mathbb{G}_n}(\pi(z), \pi(w)) \\ &\leq \inf\{k_{\mathbb{D}^n}(\widetilde{z}, \widetilde{w}) : \pi(z) = \pi(\widetilde{z}),\ \pi(w) = \pi(\widetilde{w})\} \\ &= \inf\left\{\max_{1\leq j\leq n} k_{\mathbb{D}}(\widetilde{z}_j, \widetilde{w}_j), \pi(z) = \pi(\widetilde{z}),\ \pi(w) = \pi(\widetilde{w})\right\},\ z, w \in \mathbb{D}^n. \end{aligned}$$

Consequently, $k_{\mathbb{G}_n}(p_a, q_a) \geq k_{\mathbb{D}}(a^n, -a^n) = 2k_{\mathbb{D}}(a^n, 0)$, and

$$k_{\mathbb{G}_n}(p_a, 0), k_{\mathbb{G}_n}(q_a, 0) \text{ and } k_{\mathbb{G}_n}(p_a, m_a) \leq k_{\mathbb{D}}(a, 0),$$

thus

$$\begin{aligned} &\liminf_{a\to\partial\mathbb{D}}[k_{\mathbb{G}_n}(p_a, q_a) - k_{\mathbb{G}_n}(q_a, 0) - k_{\mathbb{G}_n}(p_a, m_a)], \\ &\qquad\qquad \liminf_{a\to\partial\mathbb{D}}[k_{\mathbb{G}_n}(p_a, q_a) - k_{\mathbb{G}_n}(p_a, 0) - k_{\mathbb{G}_n}(q_a, m_a)] > -\infty, \end{aligned}$$

and finally

$$\begin{aligned} &(p_a, m_a)_0 - (p_a, q_a)_0,\ (q_a, m_a)_0 - (p_a, q_a)_0 \\ &\qquad\qquad = k_{\mathbb{G}_n}(m_a, 0) + \mathrm{O}(1) \geq c_{\mathbb{G}_n}(m_a, 0) + \mathrm{O}(1), \end{aligned}$$

where $c_{\mathbb{G}_n}$ denotes the Caratheodory distance, i.e. for a domain D, $c_D(a, b) := \sup\{k_{\mathbb{D}}(f(a), f(b)) : f \in \mathcal{O}(D, \mathbb{D})\}$. Then [**Cos2**, Corollary 3.2] provides a family $(f_\zeta) \subset \mathcal{O}(\mathbb{G}_n, \mathbb{D})$ such that for any $p \in \partial\mathbb{G}_n$, $\sup_\zeta |f_\zeta(p)| = 1$, thus $\lim_{p\to\partial\mathbb{G}_n} c_{\mathbb{G}_n}(p, 0) = \infty$, q.e.d.

The last part follows from $\mathbb{C}$-convexity of $\mathbb{G}_2$.

$\square$

Moreover,

Proposition 2.2.12. ([**N-T-T**]) *The tetrablock is not hyperbolic.*

Proof of Proposition 2.2.12 is similar to the above one and we skip it.

Buckley in [**Buc**], following Bonk, claimed that $\mathbb{D}^n$ fails to be hyperbolic because of the flatness contained in the boundary rather than the lack of smoothness that Gromov hyperbolicity fails. Recently, Gaussier and Seshadri have provided a proof of that conjecture. More precisely, the main result in [**Gau-Ses**, Theorem 1.1] states that any bounded convex domain in $\mathbb{C}^n$ whose boundary is $\mathcal{C}^\infty$-smooth and contains a non-trivial analytic disc in the boundary, is not Gromov hyperbolic with respect to the Kobayashi distance. Lemma 5.4 in their proof used the $\mathcal{C}^\infty$ assumption of smoothness in the essential way. Our aim is to prove this result in a shorter way in $\mathbb{C}^2$, assuming only $\mathcal{C}^{1,1}$-smoothness.

Theorem 2.2.13. ([**N-T-T**]) *Let D be a convex domain in $\mathbb{C}^2$ containing no complex lines.*[(2)] *Assume that ∂D is $\mathcal{C}^{1,1}$-smooth and contains an analytic disc. Then D is not Gromov hyperbolic with respect to the Kobayashi distance.*

One of the fundamental results in Gromov's theory says that for every M, $L>0$ there exists $C>0$ such that every side of any (M,L)-quasi-triangle (i.e. every side is a (M,L)-quasi-isometry) is at the distance at most C from the union of the remaining sides (cf. [**Väi**]). [**Gau-Ses**, Theorem 1.1] is based on this. Proof of Theorem 2.2.13 is based only on the estimates of the Kobayashi distance.

Proposition 2.2.14. ([**Jar-Pfl2**, Proposition 10.2.3]) *For a bounded domain G in $\mathbb{C}^2$ with smooth $\mathcal{C}^{1,1}$-boundary and a compact subset K of G there is a constant C such that*

$$k_G(z_0,z)\le -\frac{1}{2}\log d_G(z)+C,\quad \text{for } z_0\in K,\ z\in G.$$

$\mathcal{C}^2$-smoothness is assumed in [**Jar-Pfl2**] but only the locally uniform interior ball condition is used (i.e. there exists $r>0$ such that for every boundary point a of a domain G there exist a ball with radius r contained in G that is tangent to ∂G at a).

Proof of Theorem 2.2.13. Since ∂D contains an analytic disc, it is well known that it contains an affine disc (see [**N-P-Z2**, Proposition 7]). We assume that this disc has center 0 and lies in $\{z_1=0\}$, and that $D\subset\{\Re z_1>0\}$.

Lemma 2.2.15. *We can find an $r>0$ such that for any $\delta>0$ small enough there exist two discs $\mathbb{D}(\tilde p_\delta,r)$ and $\mathbb{D}(\tilde q_\delta,r)$ in $D_\delta:=D\cap\{z_1=\delta\}$ which touch ∂D at two points $\hat p_\delta$ and $\hat q_\delta$ with $|\hat p_\delta-\hat q_\delta|>5r$.*

Proof. We identify $\partial D\cap\{z_1=0\}$ with a closed, convex subset of $\mathbb{C}$, which is the closure of its interior. Call this interior D_0.

Take two different points $\hat p_0,\hat q_0\in\partial D_0$. Assume that an open segment $(\hat p_0,\hat q_0)$ is the subset of D_0. There are two possibilities. First, if $\hat p_0$ is not a $\mathcal{C}^{1,1}$-smooth boundary point of D_0. Consider a point $\tilde p$ on $(\hat p_0,\hat q_0)$ sufficiently near $\hat p_0$. Since $\tilde p\in D_0$, it has positive distance to the boundary of D_0. Choose some point where this distance

[(2)]Then D is biholomorphic to a bounded domain (cf. [**Jar-Pfl2**, Theorem 7.1.8]).

is attained, and call it $\hat{p}$. Otherwise, when $\hat{p}_0$ is $\mathcal{C}^{1,1}$-smooth, let $\hat{p} = \hat{p}_0$. Repeat the whole process for $\hat{q}_0$.

It may also happen that $(\hat{p}_0, \hat{q}_0) \not\subseteq D_0$. Then there exist discs $\mathbb{D}(\tilde{p}, r), \mathbb{D}(\tilde{q}, r) \subset D_0$ tangent to ∂D_0 at $\hat{p}$ and $\hat{q}$, where $\hat{p}, \hat{q} \in (\hat{p}_0, \hat{q}_0)$.

Now we want to move the constructed discs inside the domain. By $\mathcal{C}^{1,1}$-smoothness of D, we can move them (in $\mathbb{C}^2$) along the vector $(1, 0)$ inside D, that is $\mathbb{D}(\tilde{p}, r), \mathbb{D}(\tilde{q}, r) \subset D \cap \{z_1 = \delta\} = D_\delta$, for $0 < \delta < \delta_0$. If they do not touch ∂D_δ, then shift them (separately at every sublevel set) to the boundary but leaving their centers on the real line passing through $\tilde{p} + (\delta, 0)$ and $\tilde{q} + (\delta, 0)$. Denote new discs by $\mathbb{D}(\tilde{p}_\delta, r), \mathbb{D}(\tilde{q}_\delta, r)$, and by $\hat{p}_\delta, \hat{q}_\delta$ points of contact of those discs with ∂D_δ. □

Choose now a point $a = (\delta_0, 0) \in D$ ($\delta_0 > 0$) and consider the cone with vertex at a and base $\partial D \cap \{z_1 = 0\}$. Denote by G_δ the intersection of this cone and $\{z_1 = \delta\}$. For any $\delta > 0$ small enough the line segment with ends at $\tilde{p}_\delta$ and $\hat{p}_\delta$ intersects ∂G_δ, say at p_δ. Define q_δ in a similar way.

Set $\tilde{s}_\delta = \frac{\tilde{p}_\delta + \tilde{q}_\delta}{2}$. We shall show that $S(p_\delta, q_\delta, \tilde{s}_\delta, a) \to +\infty$ as $\delta \to 0$. For this we will see that $(p_\delta, \tilde{s}_\delta)_a - (p_\delta, q_\delta)_a \to +\infty$ as $\delta \to 0$. It will follow in the same way that $(q_\delta, \tilde{s}_\delta)_a - (p_\delta, q_\delta)_a \to +\infty$.

It is enough to prove that

$$k_D(q_\delta, a) - k_D(\tilde{s}_\delta, a) < c_1 \tag{2.2.3}$$

and

$$k_D(p_\delta, q_\delta) - k_D(p_\delta, \tilde{s}_\delta) \to +\infty. \tag{2.2.4}$$

Here and below $c_1, c_2, \ldots$ denote some positive constants which are independent of δ.

For (2.2.3), observe that Lemma 2.1.4 and Proposition 2.2.14 imply that

$$k_D(\tilde{s}_\delta, a) \geq \frac{1}{2} \log \frac{d_D(a)}{d_D(\tilde{s}_\delta)} \text{ and } 2k_D(q_\delta, a) \leq -\log d_D(q_\delta) + c_2. \tag{2.2.5}$$

It remains to use that $d_D(\tilde{s}_\delta) = d_D(q_\delta)$ for any $\delta > 0$ small enough.

To prove (2.2.4), denote by F_δ the convex hull of $\mathbb{D}(\tilde{p}_\delta, r)$ and $\mathbb{D}(\tilde{s}_\delta, r)$. Then by inclusion $k_D(p_\delta, \tilde{s}_\delta) \leq k_{F_\delta}(p_\delta, \tilde{s}_\delta)$.

Claim. $k_{F_\delta}(p_\delta, \tilde{s}_\delta) < -\frac{1}{2} \log d'_D(p_\delta) + c_3$, where d'_D is the distance to ∂D in the z_2-direction.

Indeed, for $\delta > 0$ small enough we have that

$$d'_D(p_\delta) = d_{D_\delta}(p_\delta) = d_{F_\delta}(p_\delta) = d_{\mathbb{D}(\tilde{p}_\delta, r)}(p_\delta)$$

because the closest point on ∂D_δ belongs to $\partial \mathbb{D}(\tilde{p}_\delta, r)$. Now $k_{F_\delta}(p_\delta, \tilde{s}_\delta) \leq k_{F_\delta}(p_\delta, \tilde{p}_\delta) + k_{F_\delta}(\tilde{p}_\delta, \tilde{s}_\delta)$.

Since $\mathbb{D}(\tilde{p}_\delta, r) \subset F_\delta$,

$$\begin{aligned} k_{F_\delta}(p_\delta, \tilde{p}_\delta) \leq k_{\mathbb{D}(\tilde{p}_\delta, r)}(p_\delta, \tilde{p}_\delta) &= \frac{1}{2} \log \frac{1 + \frac{|p_\delta - \tilde{p}_\delta|}{r}}{1 - \frac{|p_\delta - \tilde{p}_\delta|}{r}} \\ &\leq -\frac{1}{2} \log d_{(\tilde{p}_\delta, r)}(p_\delta) + C(r) = -\frac{1}{2} \log d'_D(p_\delta) + C(r). \end{aligned}$$

On the other hand, by using a finite chain of disks of radius r with centers on the line segment from $\tilde{p}_\delta$ to $\tilde{s}_\delta$, we obtain that

$$k_{F_\delta}(\tilde{p}_\delta, \tilde{s}_\delta) \le C\frac{|\tilde{p}_\delta - \tilde{q}_\delta|}{r} \le C(r).$$

The Claim follows.

Now, we shall show that

$$2k_D(p_\delta, q_\delta) > -\log d'_D(p_\delta) - \log d'_D(q_\delta) - c_4, \tag{2.2.6}$$

which implies (2.2.4), because $d'_D(q_\delta) \to 0$ as $\delta \to 0$.

Since the Kobayashi distance is the integrated form of the Kobayashi metric, we may find a point $m_\delta \in D$ such that

$$|p_\delta - m_\delta| = |q_\delta - m_\delta| \ge \frac{|p_\delta - q_\delta|}{2},$$
$$k_D(p_\delta, q_\delta) > k_D(p_\delta, m_\delta) + k_D(m_\delta, q_\delta) - 1.$$

Let $\check{p}_\delta \in \partial D$ be the closest point to p_δ in the direction of the complex line through p_δ and m_δ.

Recall that d'_D is the distance to ∂D in the z_2-direction and $d_D(p_\delta)$ is attained in z_1-direction for any $\delta > 0$ small enough. This means that the standard basis is adapted to the local geometry of ∂D near p_δ, and more precisely, if $X = (X_1, X_2) \in \mathbb{C}^2$ is a unit vector, [**N-P-Z2**, (4)] states in this case that there exists a constant C such that

$$\frac{1}{d_D(p_\delta, X)} \le \frac{|X_1|}{d_D(p_\delta)} + \frac{|X_2|}{d'_D(p_\delta)} \le \frac{C}{d_D(p_\delta, X)},$$

where $d_D(\cdot; X)$ is the distance to ∂D in direction X. Since $d'_D \ge d_D$, we obtain

$$d_D(p_\delta; X) \le c_5 d'_D(p_\delta).$$

Let $X := \frac{m_\delta - p_\delta}{|m_\delta - p_\delta|}$. Then $|p_\delta - \check{p}_\delta| = d_X(p_\delta)$ and thus

$$|p_\delta - \check{p}_\delta| < c_5 d'_D(p_\delta). \tag{2.2.7}$$

By convexity, D is on one of the sides, say H_δ, of the real tangent plane to ∂D at $\check{p}_\delta$. Since $\frac{|m_\delta - \check{p}_\delta|}{d_{H_\delta}(m_\delta)} = \frac{|p_\delta - \check{p}_\delta|}{d_{H_\delta}(p_\delta)}$, it follows by (2.2.5) that

$$2k_D(p_\delta, m_\delta) \ge 2k_{H_\delta}(p_\delta, m_\delta) \ge \log\frac{d_{H_\delta}(m_\delta)}{d_{H_\delta}(p_\delta)} = \log\frac{|m_\delta - \check{p}_\delta|}{|p_\delta - \check{p}_\delta|}. \tag{2.2.8}$$

Applying the triangle inequality and (2.2.7), we get that

$$\log\frac{|m_\delta - \check{p}_\delta|}{|p_\delta - \check{p}_\delta|} \ge \log\frac{|m_\delta - p_\delta| - |p_\delta - \check{p}_\delta|}{|p_\delta - \check{p}_\delta|} \ge$$
$$\log\left(\frac{r}{2|p_\delta - \check{p}_\delta|} - 1\right) \ge \log\frac{r}{2c_5 d'(p_\delta)} - 1,$$

for any $\delta > 0$ small enough. So $2k_D(p_\delta, m_\delta) > -\log d'_D(p_\delta) - c_6$. Similarly, $2k_D(q_\delta, m_\delta) > -\log d'_D(q_\delta) - c_6$, which implies (2.2.6), and completes the proof. □

Remark 2.2.16. All the above arguments hold in $\mathbb{C}^n$, $n \ge 3$, except (2.2.7).

Besides, we give a partial answer to the question raised in [**Bal-Bon**].

Theorem 2.2.17. *Let D be a $\mathcal{C}^{1,1}$-smooth convex bounded domain in $\mathbb{C}^2$ admitting a defining function of the form $\varrho(z) = -\Re z_1 + \psi(|z_2|)$ near the origin, where $\psi : [0,\varepsilon) \to \mathbb{R}_{\geq 0}$ ($\varepsilon > 0$) is a $\mathcal{C}^{1,1}$-smooth nonnegative convex function near 0 satisfying: $\psi(0) = 0$, and*

$$\limsup_{x\to 0} \frac{\log \psi(|x|)}{\log |x|} = +\infty. \tag{2.2.9}$$

Then, D is not Gromov hyperbolic.

Here one remark is worthy of notice. Before we make it we refer two definitions (cf. [**Kra1**, Chapter 11]).

• Let Ω be a domain in $\mathbb{C}^2$, $p \in \partial\Omega$. Let $\phi : \mathbb{D} \to \mathbb{C}^2$ be an analytic map (disc) such that $\phi(0) = p$ and let ρ be a $\mathcal{C}^\infty$-smooth defining function of Ω. We say that ϕ is *a nonsingular tangent disc to $\partial\Omega$ at p* if $\phi'(0) \neq 0$ and $(\rho \circ \phi)'(0) = 0$.
• (Ω, p, ρ are as above.) If there exists a nonsingular analytic disc ϕ to $\partial\Omega$ at p such that

$$|(\rho \circ \phi)(\zeta)) \leq M|\zeta|^m, \quad \text{for } |\zeta| << 1,$$

$M > 0$ is some constant, $m \in \mathbb{N}$ but does not exist a nonsingular disc satisfying the inequality with $m+1$ and some (possibly different) constant $\hat{M}$, then we say that Ω has at p *type m* or p *is of type m*. Point p *has infinite type* if p is not of type k for any $k \in \mathbb{N}$.

Under the assumption of Theorem 2.2.17 it is clear that $\phi : \mathbb{D} \to \mathbb{C}$, $\phi(\lambda) = (0,\lambda)$, $\lambda \in \mathbb{D}$ is the only nonsingular analytic disc to ∂D at 0. Observe that if ψ is $\mathcal{C}^\infty$, then 0 has infinite type if and only if the condition (2.2.9) holds.

Proof of Theorem 2.2.17. Since the case when $\psi(z_0) = 0$ for some $z_0 \neq 0$, is covered by Proposition 2.2.13, we may assume that $\psi^{-1}(0) = \{0\}$. Also assume $p = (1,0) \in D$.

Let α be an increasing function such that for any $x > 0$, $\psi'(x) \geq \psi'((1-\alpha(x))x) \geq \frac{1}{2}\psi'(x)$. We choose, for $x > 0$, $q(x) = (\psi(x), 0)$, $r(x) = (\psi(x), -(1-\alpha(x))x)$, $s(x) = (\psi(x), (1-\alpha(x))x)$.

We claim that:

(1) $d_D(q(x)) = \psi(x)$, $x \in (0,\varepsilon)$,
(2) $\frac{\alpha(x)}{4} x\psi'(x) \leq d_D(r(x)), d_D(s(x)) \leq \alpha(x)x\psi'(x)$, $x \in (0,\varepsilon)$,
(3) the functions $k_D(s(\cdot), q(\cdot)) + \frac{1}{2}\log\alpha(\cdot)$ and $k_D(r(\cdot), q(\cdot)) + \frac{1}{2}\log\alpha(\cdot)$ are bounded on $(0,\varepsilon)$,
(4) the function $k_D(r(\cdot), s(\cdot)) + \log\alpha(\cdot)$ is bounded on $(0,\varepsilon)$.

Before we proceed to prove the claims we make some general observation about the infinite order of vanishing.

Lemma 2.2.18. *For any $\hat{\varepsilon} \in (0,\varepsilon)$ and $A > 0$, there exists $x \in (0,\hat{\varepsilon})$ such that $\frac{x\psi'(x)}{\psi(x)} > A$.*

Proof. Suppose instead that there exist $\hat{\varepsilon} > 0$ and $A > 0$ such that $\frac{x\psi'(x)}{\psi(x)} \leq A$ for $0 < x \leq \hat{\varepsilon}$. Then

$$\frac{d}{dx}\left(\log \psi(x)\right) \leq \frac{A}{x}, \quad 0 < x \leq \hat{\varepsilon},$$

so $\log(\psi(\varepsilon)) - \log(\psi(x)) \leq A\left(\log \varepsilon - \log x\right)$, i.e.

$$\psi(x) \geq \frac{\psi(\varepsilon)}{\varepsilon^A} x^A, \quad 0 < x \leq \hat{\varepsilon},$$

which means that at the point 0 there is finite order of contact with the tangent hyperplane, a contradiction. $\square$

Assume the claims for a while, and observe that for any $x \in (0, \epsilon)$ we have

$$(r(x), p)_{q(x)} - (r(x), s(x))_{q(x)}, \quad (p, s(x))_{q(x)} - (r(x), s(x))_{q(x)} \geq -\frac{1}{2}\log\frac{\psi(x)}{x\psi'(x)} + C_1.$$

Since the above quantities can be made arbitrarily large, it finishes the proof.

It remains to prove (1)-(4). Fix $x \in (0, \varepsilon)$.

(1) is clear. Next, since $(\psi((1-\alpha(x))x), (1-\alpha(x))x) \in D$, $d_D(s) \leq \psi(x) - \psi((1-\alpha(x))x)) \leq \alpha(x)x\psi'(x)$ by convexity. Let L be the real line through $(\psi((1-\alpha(x))x), (1-\alpha(x))x)$ and $(\psi(x), x)$. Its slope is less than $\psi'(x)$, so $d_D(s(x)) \geq d_{L'}(s(x))$, where L' is the line through $(\psi((1-\alpha(x))x), (1-\alpha(x))x)$ with slope $\psi'(x)$, so

$$d_D(s(x)) \geq \frac{\psi(x) - \psi((1-\alpha(x))x)}{\sqrt{1+\psi'(x)^2}}$$
$$\geq \frac{1}{2}\alpha(x) \times \psi'((1-\alpha(x))x) \geq \frac{1}{4}\alpha(x) \times \psi'(x).$$

Thus, $\frac{\alpha(x)}{4}x\psi'(x) \leq d_D(s(x)) \leq \alpha(x)x\psi'(x)$. Analogous estimates hold for $r(x)$, which gives (2).

The analytic disc $\zeta \mapsto (\psi(x), x\zeta)$ provides immediate upper estimates in (3) and (4).

To get the lower estimate for the function $k_D(s(\cdot), q(\cdot))$, we map D to a domain in $\mathbb{C}$ by the complex affine projection π_s to $\{z_1 = \psi(x)\}$, parallel to the complex tangent space to ∂D at the point $(\psi(x), x)$. Then $\pi_s(D) = \{\psi(x)\} \times D_s$, where D_s is a convex domain in $\mathbb{C}$, containing the disk $\{|z_2| < x\}$, and its tangent line at the point x is the real line $\{\Re z_2 = x\}$. The projection is given by the explicit formula

$$\pi_s(z_1, z_2) = \Big(\psi(x), z_2 + \frac{\psi(x) - z_1}{\psi'(x)}\Big), \ (z_1, z_2) \in \mathbb{C}^2.$$

We renormalize π_s by setting $f_+(z) := 1 - \frac{1}{x}[\pi_s(z)]_2, \ z \in \mathbb{C}^2$. Therefore $f_+(D) \subset H := \{z \in \mathbb{C} : \Re z > 0\}$, so

$$k_D(s(x), q(x)) \geq k_H(f_+(s(x)), f_+(q(x)) = k_H(\alpha(x), 0) \geq -\frac{1}{2}\log\alpha(x) + C_2, \quad (2.2.10)$$

where $C_2 > 0$ does not depend on x.

The estimate for $k_D(r(x), q(x))$ proceeds along the same lines, but we use the projection π_r to $\{z_1 = \psi(x)\}$ along the complex tangent space to ∂D at $(\psi(x), x)$, given by

$$\pi_r(z_1, z_2) = \Big(\psi(x), z_2 - \frac{\psi(x) - z_1}{\psi'(x)}\Big), \ (z_1, z_2) \in \mathbb{C}^2.$$

Note that choosing $f_-(z) = 1 + \frac{1}{x}[\pi_r(z)]_2$, $z \in \mathbb{C}^2$ we have $f_-(D) \subset \{w : \Re w > 0\}$.

Now we tackle the lower estimates for $k_D(r(x), s(x))$. Let $\gamma : [0, 1] \to D$ be any piecewise $\mathcal{C}^1$ curve such that $\gamma(0) = s(x)$, $\gamma(1) = r(x)$. Let $c_0 \in (\alpha(x), \frac{1}{2})$. We claim that there exists $t_0 \in (0, 1)$ such that if we set $u = \gamma(t_0)$, then $|f_+(u)|, |f_-(u)| \geq c_0$.

For this write $\gamma = (\gamma_1, \gamma_2)$. Set $\zeta_1(t) := 1 - \frac{\psi(x) - \gamma_1(t)}{x\psi'(x)}$, $t \in [0, 1]$. By the explicit form of π_s, the condition $|f_+(u)| \geq c_0$ reads $|\zeta_1 - \frac{\gamma_2(t_0)}{x}| \geq c_0$, and the condition $|f_-(u)| \geq c_0$ reads $|\zeta_1 + \frac{\gamma_2(t_0)}{x}| \geq c_0$ (we are searching for a good candidate of t_0, we only wrote the conditions which t_0 must satisfy). We claim that the disks $\overline{\mathbb{D}}(\zeta_1(t), c_0)$ and $\overline{\mathbb{D}}(-\zeta_1(t), c_0)$ are disjoint for any t. Indeed, they would intersect for some $\hat{t}$ if and only if $0 \in \overline{\mathbb{D}}(\zeta_1(\hat{t}), c_0)$, which implies

$$\Re\Big(\frac{\gamma_1(\hat{t})}{x\psi'(x)}\Big) \leq -1 + c_0 + \frac{\psi(x)}{x\psi'(x)} \leq -\frac{1}{3}$$

for any x such that $\frac{\psi(x)}{x\psi'(x)} \leq \frac{1}{6}$, which we may assume by Lemma 2.2.18. In particular $\Re\gamma_1(\hat{t}) < 0$, the contradiction (with the assumptions about D).

Now let $t_1 := \max\{t : \frac{\gamma_2(t)}{x} \in \overline{\mathbb{D}}(\zeta_1(t), c_0)\}$. Then $\frac{\gamma_2(t_1)}{x} \notin \overline{\mathbb{D}}(-\zeta_1(t_1), c_0)$, and by continuity there is $\eta > 0$ such that $\frac{\gamma_2(t_1+\eta)}{x} \notin \overline{\mathbb{D}}(-\zeta_1(t_1 + \eta), c_0)$, and of course $\frac{\gamma_2(t_1+\eta)}{x} \notin \overline{\mathbb{D}}(\zeta_1(t_1 + \eta), c_0)$ by maximality of t_1. So $t_0 = t_1 + \eta$ will provide a point satisfying the claim.

Consequently, taking a curve γ such that

$$k_D(r(x), s(x)) + 1 \geq \int_0^1 \kappa_D(\gamma(t), \gamma'(t))dt,$$

$$\begin{aligned}\int_0^1 \kappa_D(\gamma(t), \gamma'(t))dt = \int_0^{t_0} \kappa_D(\gamma(t), \gamma'(t))dt + \int_{t_0}^1 \kappa_D(\gamma(t), \gamma'(t))dt \\ \geq k_D(r(x), u) + k_D(u, s(x)),\end{aligned}$$

and proceeding as in (2.2.10) we end the proof.

□

There naturally arises the question whether there is any connection between Gromov hyperbolicity and pseudoconvexity. The known examples do not say anything in this matter. Also, it is easy to construct domains which are Gromov hyperbolic but neither pseudoconvex nor smooth. Indeed, take any strictly pseudoconvex domain G. Assume that A is a closed subset of G with $H^{2n-2}(A) = 0$, where H^{2n-2} denotes the $(2n - 2)$-dimensional Hausdorff measure. To prove our claim, note that

$$k_G|_{(G\setminus A)\times(G\setminus A)} = k_{G\setminus A}$$

(for details see [**Jar-Pfl2**, Theorem 3.4.2] and Hartogs Theorem in Appendix). Now it remains to apply Theorem 1.5.16.

The above example does not have a smooth boundary. The next proposition yields, in particular, a family of non pseudoconvex domains with smooth boundaries which are Gromov hyperbolic.

Proposition 2.2.19. *Let $G \subset \mathbb{C}^n$, $n \geq 2$ be a bounded strictly pseudoconvex domain and let $D \Subset G$ have one of the following form:*

- *D is $\mathcal{C}^2$-smooth and its Levi form has at least one strictly positive eigenvalue at each boundary point and $G \setminus \overline{D}$ is a domain.*
- *D is a polydisc.*

Then $G \setminus \overline{D}$ is Gromov hyperbolic.

In the last proof of Section 2.2.20 we utylize already mentioned result by Balogh-Bonk.

Theorem 2.2.20. [**Bal-Bon**] *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$ ($n \geq 2$). Then (Ω, k_Ω) is Gromov hyperbolic.*

The Reader might have noticed how the work by Zoltán M. Balogh and Mario Bonk plays a prominent role in the thesis. Here, the author would like to express her admiration for both mathematicians.

Proof of Proposition 2.2.19. First, assume that the Levi form of D at any boundary point has at least one positive eigenvalue. Recall that [**Jar-Pfl3**, Proposition 2.2.3(c)]) implies that every defining function of D has at least one positive eigenvalue.

Since, by [**Bal-Bon**], every stricty pseudoconvex domain is Gromov hyperbolic with respect to the Kobayashi distance, it remains to be shown the boundedness of the function $k_{G\setminus\overline{D}} - k_G$ on $(G \setminus \overline{D}) \times (G \setminus \overline{D})$.

Near the inner boundary, it follows from the estimates for the Kobayshi metric in [**Kra1**] and [**For-Lee**] for $\mathbb{C}^2$ and $\mathbb{C}^n$, respectively. Recall

$$\kappa_{G\setminus\overline{D}}(z, \nu_z) \approx d_{G\setminus\overline{D}}(z)^{-\frac{3}{4}}, \tag{2.2.11}$$

where ν_z is the outside normal vector at point that is in ∂D ad is the closest one to z.

Let $D^\varepsilon := \{z \in G \setminus \overline{D} : d_D(z) \leq \varepsilon\}$. For $\varepsilon > 0$ small enough, $D^\varepsilon \Subset G$, and $D^\varepsilon = \{\zeta + t\nu_\zeta : 0 < t \leq \varepsilon\}$, where ν_ζ is the outside unit normal vector to ∂D at ζ.

Let $K^\varepsilon := \{z \in G \setminus \overline{D} : d_D(z) = \varepsilon\}$. It is a compact subset of $G \setminus \overline{D}$ on which $k_{G\setminus\overline{D}} - k_G$ is clearly bounded. By (2.2.11), the Kobayashi distance $k_{G\setminus\overline{D}}$ between any point in D^ε and any point in K^ε is bounded from above. Obviously, the k_G distance, too. So, the difference between those two distances cannot become unbounded near the inner boundary.

We proceed to investiage k_G and $k_{G\setminus\overline{D}}$ near ∂G. Fix two sequences $\{z_\mu\}$, $\{w_\mu\} \subset G \setminus \overline{D}$. We show that the sequence $k_{G\setminus\overline{D}}(z_\mu, w_\mu) - k_G(z_\mu, w_\mu)$ is bounded and then use the compactness of ∂G. Without loss of generality, we may assume that $z_\mu \to z, w_\mu \to w,\ z, w \in \overline{G} \setminus D^\varepsilon$.

Let us first consider case when $z \neq w$. If z or w is in $G \setminus \overline{D}$ then the sequence $k_{G\setminus\overline{D}}(z_\mu, w_\mu) - k_G(z_\mu, w_\mu)$ is bounded. Indeed, [**For-Ros**, Theorem 2.3] says that the estimate from below in the spirit of Proposition 2.2.14 holds for $k_{G\setminus\overline{D}}$ as well as for k_G.

The same conclusion we obtain when both z and w are boundary points of G if we apply [**For-Ros**, Corollary 2.4, Proposition 2.5].

The remaining situation, i.e. when $z = w \in \overline{G} \setminus G$, follows from Theorem 1.5.17 and Theorem 2.2.20. Indeed, Theorem 1.5.17 shows that the behavior of the Kobayashi distance has a local character. In other words, in a neighborhood of ∂G distances k_G and $k_{G\setminus\overline{D}}$ depends on ∂G which is common for G and $G \setminus \overline{D}$. And it just remains to apply Theorem 2.2.20.

For the case where D is a polydisc, since all the distances considered are holomorphically invariant, we might assume that $D = \mathbb{D}^n$. All the above arguments might be repeated except now we do not know the behavior of Kobayashi metric near the inner boundary. However, we might proceed as follows. Let $r > 0$ be such that $(1+r)\overline{\mathbb{D}}^n \subset G$. Take some $z^0 \in [(1+\varepsilon)\mathbb{D}^n] \setminus \overline{\mathbb{D}}^n$, where $0 < 3\varepsilon < r$. Thus, $1+\varepsilon \geq |z_j^0| > 1$ for some $1 \leq j \leq n$. Observe that

$$\inf_{z\in\partial G, w\in\partial((1+\varepsilon)\mathbb{D}^n)} |z-w| > 2\varepsilon.$$

Choose any point $w(z^0) \in \{z_j = z_j^0\} \cap \partial((1+2\varepsilon)\mathbb{D}^n)$, which realizes the distance from z^0 to $\partial((1+2\varepsilon)\mathbb{D}^n) \cap \{z_j = z_j^0\}$. Let $\Omega_\varepsilon := \{z_j = z_j^0, |z_k| < 1+3\varepsilon \text{ for } k \neq j\} \subset \mathbb{C}^{n-1}$. With our choices, we have

$$\begin{aligned} &k_{G\setminus\overline{D}}(z^0, w(z^0)) \leq \\ &\qquad \sup\left\{k_{\Omega_\varepsilon}(z,w) :\ z_j = w_j = z_j^0;\ |z_k| \leq 1+\varepsilon, |w_k| \leq 1+2\varepsilon, k \neq j\right\} < M, \end{aligned}$$

for some finite number M which does not depend on z^0. The connectedness of $\partial((1+2\varepsilon)\mathbb{D}^n)$ ends the proof.

□

2.3. The Carathéodory metric for the symmetrized bidisc

In [**Agl-You1**] the Authors computed the Carathéodory distance at the origin. So, one can easily find the Carathéodory-Reiffen metric in 0. Recall, if (s_1, s_2) is a point from $\mathbb{G}_2$ then

$$c_{\mathbb{G}_2}((0,0),(s_1,s_2)) = \frac{2|s_1 - s_2\overline{s}_1| + |s_1^2 - 4s_2|}{4 - |s_1|^2}$$

(see [**Agl-You1**]). Consequently by taking the limit

$$\gamma_{\mathbb{G}_2}((0,0);(X_1,X_2)) = \frac{|X_1| + 2|X_2|}{2}, \tag{2.3.1}$$

where $(X_1, X_2) \in \mathbb{C}^2$ (cf. e.g. [**Jar-Pfl2**, Proposition 2.5.1]).

A formula for $\gamma_{\mathbb{G}_2}$ at arbitrary point was derived independently by Costara ([**Cos1**]), and Agler & Young ([**Agl-You1**]). For $p \in [0,1)$, we have

$$\begin{aligned} &\gamma_{\mathbb{G}_2}\big((0,p);X\big) \\ &\qquad = \max\Big\{\gamma_{\mathbb{D}}\big(f_\omega(0,p); f_\omega'(0,p)(X)\big) : f_\omega(x,y) = \frac{2\omega y - x}{2 - \omega x}, \omega \in \mathbb{T}\Big\}. \end{aligned} \tag{2.3.2}$$

But

$$\gamma_{\mathbb{D}}\big(f_\omega(0,p); f'_\omega(0,p)(X)\big) = \frac{\big|\omega p X_1 + 2X_2 - \overline{\omega} X_1\big|}{2(1-p^2)}. \tag{2.3.3}$$

It implies that

$$\gamma_{\mathbb{G}_2}((0,p);(X_1,X_2)) = \gamma_{\mathbb{G}_2}((0,p);(X_1,\overline{X}_2)) = \gamma_{\mathbb{G}_2}((0,p);(X_1,-X_2)),$$

for $X_1 \geq 0,\ 1 > p \geq 0$. Thus, if $X_2 = r_2 e^{i\phi}$ and $r_2 \geq 0$, we can assume that $\phi \in [0, \frac{\pi}{2}]$.

For any positive real numbers a, b, consider the equation

$$H(\lambda) = \lambda^4 - \lambda^2(2 + a^2 + b^2) + 2\lambda(a^2 - b^2) + (1 - a^2 - b^2) = 0. \tag{2.3.4}$$

Note, (2.3.4) has only one solution in $(-\infty,-1)$. Indeed, let us define $G(\lambda) = \frac{a^2}{(\lambda+1)^2} + \frac{b^2}{(\lambda-1)^2}$. If $\lambda \in (-\infty,-1)$, then $G(\lambda) = 1$ iff $H(\lambda) = 0$.

To formulate the next lemma, we shall need some auxiliary constants

$$a = \frac{r_2 \sin\phi\,(p+1)}{pr_1}, \quad b = \frac{r_2\cos\phi\,(1-p)}{pr_1}, \tag{2.3.5}$$

where $r_1 > 0,\ r_2 \geq 0,\ 1 > p > 0,\ \frac{\pi}{2} \geq \phi \geq 0$.

Lemma 2.3.1. *Let $p \in (0,1)$ and $X = (X_1, X_2) = (r_1, r_2 e^{i\phi}) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}$. For $r_1 r_2 \neq 0$ let a, b be the constants given by (2.3.5), and let λ be the only root of the polynomial (2.3.4) in $(-\infty,-1)$. Then,*

$$\gamma_{\mathbb{G}_2}\big((0,p);X\big) = \begin{cases} \dfrac{\sqrt{(p+1)^2|X_1|^2 + (4+\frac{(1-p)^2}{p})|X_2|^2}}{2(1-p^2)} & \\ \qquad\qquad \text{if } pr_1r_2 \neq 0,\ \sin\phi = 0,\ b \leqslant 2, & \\ \dfrac{\sqrt{[1+p^2-2p(2\lambda+1)+\frac{4pb^2}{(1-\lambda)^2}]|X_1|^2 + 4|X_2|^2}}{2(1-p^2)} & \\ \qquad\qquad \text{if } pr_1r_2 \neq 0,\ \sin\phi \neq 0, \text{ or} & \\ \qquad\qquad \text{if } pr_1r_2 \neq 0,\ \sin\phi = 0,\ b \geqslant 2. & \end{cases}$$

Remark 2.3.2. Cases not covered by the Lemma 2.3.1 can be achieved by considering the relevant limits (recall $\gamma_{\mathbb{G}_2}$ is locally Lipschitz - cf. [**Jar-Pfl2**, Proposition 2.5.1]).

Proof. From (2.3.3)

$$[2(1-p^2)\gamma_{\mathbb{G}_2}((0,p);X)]^2 = r_1^2(p^2+1) + 4r_2^2 + 2pr_1^2\Big[\big(\sin\theta + a\big)^2 - \big(\cos\theta + b\big)^2 - a^2 + b^2\Big]$$

($\omega = e^{i\theta}$). In this way, we reduced the problem to finding the global maximum of the function $f(x,y) = (x+a)^2 - (y+b)^2$ on the unit circle $x^2+y^2 = 1$. The method of Lagrange multipliers gives the solution. □

Remark 2.3.3. From Lemma 2.3.1 we may deduce that the Carathéodory metric is not differentiable (of course, the intresting case is $X \neq 0$). Indeed, the differentiability is lost at points $((0,p);(1,r_2))$, where $p \in (0,1)$, and r_2 is a positive real number such that b in (2.3.5) is equal to 2. For this purpose, it is enough to consider the limits:

$$\lim_{\mathbb{R}\ni X_2\to r_2^+}\frac{\gamma_{\mathbb{G}_2}((0,p);(1,X_2))-\gamma_{\mathbb{G}_2}((0,p);(1,r_2))}{X_2-r_2}=Cp\Big(\frac{2(1-p)}{p}+\frac{4}{1-p}\Big),$$

$$\lim_{\mathbb{R}\ni X_2\to r_2^-}\frac{\gamma_{\mathbb{G}_2}((0,p);(1,X_2))-\gamma_{\mathbb{G}_2}((0,p);(1,r_2))}{X_2-r_2}=Cp\Big(\frac{1-p}{p}+\frac{4}{1-p}\Big),$$

$C=\frac{1}{2(1-p^2)^2\gamma_{\mathbb{G}_2}((0,p);(1,r_2))}$.

Remark 2.3.3 provides the example of a $\mathbb{C}$-convex domain on which the Lempert function coincides with the Carathéodory distance but whose the Kobayashi (or in this case also the Carathéodory) metric is not differentiable.

Appendix

In this Chapter we collect helpful information which we applied in different places in the dissertation.

Information from functional analysis

We recommend very elementary and, in our opinion, very original aproach to functional analysis by Barbara MacCluer [**MaC**].

Theorem 1. Riesz Theorem (cf. [**MaC**, pg.17])

Every bounded linear functional Λ *on a Hilbert space* $(\mathcal{H}, \langle\ ,\ \rangle)$ *is given by an inner product with a (unique) fixed vector* h_0 *in* $\mathcal{H}$*:* $\Lambda(h) = \langle h, h_0 \rangle$*. Moreover, the norm of the linear functional* Λ *is* $\|h_0\|$*.*

Definition 1. Adjoint operator (cf. [**MaC**, pg.34])

Assume that $(H, \langle\ ,\ \rangle_H)$ *and* $(K, \langle\ ,\ \rangle_K)$ *are Hilbert spaces. Let* $A : H \to K$ *be a bounded linear operator. An adjoint of* A *is a (unique) bounded linear operator* $A^* : K \to H$ *such that*

$$\langle Ah, k \rangle_K = \langle h, A^* \rangle_H, \quad h \in H,\ k \in K.$$

Information from complex analysis

All facts stated below (with proofs) the reader might find in: [**Gun**], [**Jak-Jar**], [**Jar-Pfl1**], [**Jar-Pfl2**], [**Kra2**], [**Rud2**].

Definition 2. Holomorphic map

Let D *be a domain in* $\mathbb{C}^n$ *and let* $F : D \to \mathbb{C}$*. If for every point* $p \in D$ *there exist a radius* $r > 0$ *and a sequence* $\{a_\alpha\}_{\alpha \in \mathbb{N}^n} \subset \mathbb{C}$ *such that*

$$F(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - p)^\alpha, \ z \in \mathbb{D}^n(p, r).$$

Theorem 2. Hartogs Theorem (cf. [**Kra2**, Section 2.4])

Let $D \subset \mathbb{C}^n$ *be a domain and let* $F : D \to \mathbb{C}$*. If for every* $a = (a_1, \ldots, a_n) \in D$ *and* $1 \le j \le n$ *the function*

$$F(a_1, \ldots a_{j-1}, \cdot, a_{j+1}, \ldots, a_n)$$

is holomorphic on the set

$$\{z \in \mathbb{C} : (a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_n) \in D\},$$

then F is holomorhic on D.

Definition 3. Pseudoconvex set ([**Jak-Jar**, Chapter 2, Chapter 4])

Let D be a non empty domain in $\mathbb{C}^n$.

We call D a domain of holomorphy if the following condition is satisfied. If there are no domains $\tilde{D}$ and D_0 such that

(1) $\tilde{D} \not\subseteq D$,

(2) $\varnothing \subsetneq D_0 \subset D \cap \tilde{D}$,

(3) for every $f \in \mathcal{O}(D)$ there exists an $\tilde{f} \in \mathcal{O}(\tilde{D})$ such that $f = \tilde{f}$ on D_0.

The solution of the Levi problem says that D is a domain of holomorphy if and only if D is *pseudoconvex*.

The original definition of a pseudoconvex set and its equivalence with the above definition the Reader might find in every classical book on complex analysis, e.g. [**Jak-Jar**], [**Jar-Pfl3**], [**Kra2**].

Definition 4. Strictly pseudoconvex domain (Ibid.)

We say that a domain D is strictly pseudoconvex if D is bounded, has $\mathcal{C}^2$ smooth boundary and the Levi form $\mathcal{L}_\rho$ of the defining function ρ of D is positive defined, i.e.

$$\mathcal{L}_\rho(a;X) = \sum_{k,l=1}^{n} \frac{\partial^2 \rho(a)}{\partial z_k \partial \overline{z}_l} X_k \overline{X}_l > 0, \quad a \in \partial D, \ X \in \mathbb{C}^n \setminus \{0\}. \tag{2.3.6}$$

If the condition (2.3.6) *holds only for points near a_0 then we say that D is strictly pseudoconvex at a_0.*

Theorem 3. (Ibid.)

Every strictly pseudoconvex domain is pseudoconvex.

Theorem 4. ([**Jar-Pfl2**, pg. 185-186])

Let $D \subset \mathbb{C}^n$ be a bounded domain. Then β_D the Bergman metric on D is positively defined, i.e. $\beta_D(a;X) > 0, a \in D, \ X \in (\mathbb{C}^n)_*$.

Definition 5. *Let X and Y be topological spaces. A continuous map $f : X \to Y$ is said to be proper if $f^{-1}(K)$ is a compact set in X for every compact $K \subset X$.*

Theorem 5. Basic properties of proper holomorphic mappings (cf. [**Rud2**, Chapter 15])

Let $D \subset \mathbb{C}^n$ be a domain. Assume that $F : D \to \mathbb{C}^n$ is a proper holomorphic mapping onto the image. Then:

- *F is a closed map.*
- *$F(D) =: G$ is an open set.*

Let $M := F(\{x \in D : JF(x) = 0\})$.

- *Set $G \setminus M$ is a connected open set that is dense in G. Set $G \setminus M$ (M) is called a regular* (respectively *critical) set of F and its element is called regular* (respectively *critical) value of F.*

• *For every* $x \in G$ *the set* $F^{-1}(x)$ *is finite.*
Let $\#(x) :=$ *the number of points in set* $F^{-1}(x)$, $x \in G$.
• *There is an integer* m (*the so-called multiplicity of* F) *such that*

$$\#(x) = m \text{ for every regular value of } F,$$
$$\#(x) < m \text{ for every critical value of } F.$$

• M *is an analytic set, i.e. it is locally a zero set of some non-constant analytic function.*

Theorem 6. Riemann Riemovable Singularity Theorem (cf. [**Jar-Pfl2**, Theorem 4.2.9])
Let D *be any domain in* $\mathbb{C}^n$ *and let* A *be an analytic subset of* D. *Then*

$$\mathbb{A}^2(D \setminus A) = \mathbb{A}^2(D)|_{D\setminus A}.$$

Theorem 7. Hartogs Theorem (cf. [**Kra2**, pg. 33])
Let $D \subset \mathbb{C}^n$ *a bounded domain,* $n > 1$. *Let* K *be a compact subset of* D *with the property that* $D \setminus K$ *is connected. If* f *is holomorphic on* $D \setminus K$, *then there is a holomorphic* F *on* D *such that* $F|_{D\setminus K} = f$.

Definition 6. Cartan domain of second type $\mathcal{R}_{II}$ in $\mathbb{C}^3$ (cf. [**Hua**])
Define

$$\mathcal{R}_{II} := \{\widetilde{z} \in \mathcal{M}_{2\times 2}(\mathbb{C}) : \widetilde{z} = \widetilde{z}^t,\ \|\widetilde{z}\| < 1\},$$

where $\|\cdot\|$ *is the operator norm and* $\mathcal{M}_{2\times 2}(\mathbb{C})$ *denotes the space of* 2×2 *complex matrices* (*we identify a point* $(z_{11}, z_{22}, z) \in \mathbb{C}^3$ *with a* 2×2 *symmetric matrix* $\begin{pmatrix} z_{11} & z \\ z & z_{22} \end{pmatrix}$).
The set $\mathcal{R}_{II}$ *is called the Cartan domain of second type in* $\mathbb{C}^3$.
The Bergman kernel function of $\mathcal{R}_{II}$ (*after the identification*) *is given by a formula*

$$K_{\mathcal{R}_{II}}(z, w) = \det(I - z\overline{w})^{-3},\ z,\ w \in \mathcal{R}_{II}.$$

Recall that $\mathrm{Vol}(\mathcal{R}_{II}) = \frac{\pi^3}{6}$.

Definition 7. Hartogs domain (cf. [**Jak-Jar**, Section 1.6])
Let D *be a domain in* $\mathbb{C}^n$, *let* $1 \le k \le n-1$, *and let* G *denote the projection of* D *onto* $\mathbb{C}^{n-k}$, $G = \pi(D)$, *where*

$$\pi : \mathbb{C}^{n-k} \times \mathbb{C}^k \ni (z, w) \to z \in \mathbb{C}^{n-k}.$$

We say that D *is a Hartogs domain over* G *if for any* $z \in G$ *the fiber* $D_z = \{w \in \mathbb{C}^k : (z, w) \in D\}$ *is balanced.*

List of symbols

General symbols

$\mathbb{N}$ - the set of natural numbers: $1, 2, 3, \ldots$
$\mathbb{C}$ - the field of complex numbers
$\mathbb{R}$ - the field of real numbers
$\mathbb{R}_{>0} := \{t \in \mathbb{R} : r > 0\}$
$\mathbb{R}_{\geq 0} := \{t \in \mathbb{R} : r \geq 0\}$
$A_* := \{x \in A : x \neq 0\}$
$\Re$ - the real part
$\Im$ - the imaginary part
$a \cdot (q_1, \ldots, q_n) = (aq_1, \ldots, aq_n)$
$\overline{w}$ - the conjugate of $w,\ w \in \mathbb{C}$
$\langle z, w\rangle := \sum_{j=1}^{k=n} z_j \overline{w}_j,\ z = (z_1, \ldots, z_n),\ w = (w_1, \ldots, w_n) \in \mathbb{C}^n$
$\langle\ ,\ \rangle$ - the Hermitian scalar product in $\mathbb{C}^n$
$A^\perp := \{x : \langle x, a\rangle = 0$ for $a \in A\}$
$|z| = |z|_n =:= (\langle z, z\rangle)^{1/2} = (|z_1|^2 + \ldots + |z_n|^2),\ z \in \mathbb{C}^n$
$|\ |$ - the Euclidean norm in $\mathbb{C}^n$
$\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\},\ a \in \mathbb{C},\ r > 0$
$\mathbb{D} := \mathbb{D}(0, 1)$ - the unit disc
$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ - the unit circle
$\mathbb{D}^n(p, r) := \mathbb{D}(p_1, r_1) \times \ldots \times \mathbb{D}(p_n, r_n),\ p = (p_1, \ldots, p_n) \in \mathbb{C}^n,\ r =$
$\mathbb{D}^n(p, r) := \mathbb{D}^n(p, (r, \ldots, r))\ (a \in \mathbb{C}^n,\ r \in \mathbb{R})$
$(r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$
$B_d(a, r)$ - the ball with center at a point a and radius $r > 0$ with respect to a distance d
$B_d(a, r) = B(a, r) = \{z : d(a, z) < r\}$
$\mathbb{B} = \mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\} \subset \mathbb{C}^n$
$\mathbb{B}$ - the Euclidean unit ball in $\mathbb{C}^n$
$dV = dV_n$ - $2n$ dimensional Lebegue measure
l (the Euclidean) length of a curve
$d(a, A) := \inf\{d(a, b) : b \in A\}$, (X, d) metric space, $a \in X$, $\emptyset \neq A \subset X$
$\mathrm{Vol}(D)$ - the volume of a set D (with respect to the Lebegue measure)
$\mathcal{C}(D, G)$ - the space of all continous function $F : D \to G$
$\mathcal{C}^k(D, G)$ - the space of all $\mathcal{C}^k$-mappings $F : D \to G,\ k \in \mathbb{N} \cup \{\infty, \omega\}$

$\mathcal{C}^{n,\alpha}(D)$ - the space of all n times continuously differentiable functions $f : D \to \mathbb{C}$ such that $|f^{(n)}(x) - f^{(n)}(y)| \leq M|x-y|^{\alpha}$ for some $M > 0$ and all $x, y \in D \subset \mathbb{R}^k$ ($k, n \in \mathbb{N},\ \alpha > 0$, D is an open set)

γ^* - the image of γ

$\mathcal{O}(\Omega_1, \Omega_2)$ - the space of all holomorphic mappings $F : \Omega_1 \to \Omega_2$

$\mathcal{O}(\Omega) = \mathcal{O}(\Omega, \mathcal{C}),\ \Omega \subset \mathbb{C}^n$

$J\pi$ - the complex Jacobian of a map π

$\lfloor\ \rfloor$ - the greatest integer function

$\mathcal{S}_n$ - the group of all permutations of a set $\{1, 2, \ldots, n\}$

$\|\ \|$ - the norm of a functional

$\mathbb{I}_A$ - the identity operator on A

A^* - the (Hilbert space) adjoint of A

$\mathcal{M}_{n\times n}$ - the space of $n \times n$ square matrices

$d_D(z) := \inf\{d(z,x) : x \in \partial D\},\ D \subset X,\ (X,d)$ metric space

d_D - the distance function

$\mathcal{L}_\rho(z;X) = \sum_{1\leq k,\,l\leq n} \frac{\partial^2 \rho}{\partial z_k \partial \overline{z}_l} X_k \overline{X}_l$, where $\rho : D \to \mathbb{C}$ is a $\mathcal{C}^2$ function on an open set $D \subset \mathbb{C}^n$, $z \in D$, $X \in \mathbb{C}^n$

$\mathcal{L}_\rho$ - the Levi form of ρ

$f^+ := \max\{f, 0\}$

$C(n,k)$ - the binomial coefficient

Chapter 1

Chapter 2

References

[A-W-Y] A.A. Abouhajar, M.C. White, N.J. Young, *A Schwarz lemma for a domain related to μ-synthesis*, J. Geom. Anal., 17(2007), 717-750.

[A-Y-Y] J. Agler, F.B. Yeh, N.J. Young, *Realization of functions into the symmetrized bidisc* in Reproducing Kernel Spaces and Applications, ed. D. Alpay, Operator Theory: Advances and Applications 143, Birkhäuser Verlang (2003), 1-37.

[Agl-You1] J. Agler, N.J. Young, *A Schwarz lemma for the symmetrized bidisc*, Bull. London Math. Soc., 33(2001), 175-186.

[Agl-You2] J. Agler, J. Young, *The two-by-two spectral Nevanlinna-Pick problem*, Trans. Amer. Math. Soc., 356(2003), 573-585.

[Ala] G. Aladro, *The comparability of the Kobayashi approach region and the admissible approach region*, Illinois J. Math., 33 (1989), 42-63.

[A-P-S] M. Andersson, M. Passare, R. Sigurdsson, *Complex convexity and analytic functionals*, Birkhäuser, Basel-Boston-Berlin, 2004.

[Aro] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., 68(3)(1950), 337-404.

[B-M-V] G.P. Balakumar, P. Mahajan, K. Verma, *Bounds for invariant distances on pseudoconvex Levi corank one domains and applications*, arXiv:1303.3439.

[Bal-Bon] Z.M. Balogh, M. Bonk, *Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains*, Comment. Math. Helv., 75 (2000), 504-533.

[Bell] S.R. Bell, *The Bergman kernel function and proper holomorphic mappings*, Trans. Amer. Math. Soc., 270(1982), 685-691.

[Bella] A. Bellaïche, *The tangent space in sub-Riemannian geometry* in Sub-Riemannian geometry Eds A. Bellaïche, J.-J. Risler, Progress in Math., 144, Bierkhäuser, Basel, 1996, 1-78.

[Ber1] S. Bergman, *Über die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach Orthogonal funktionen*, Math. Annalen, 86(1922), 238-271.

[Ber2] S. Bergman, *The Kernel Function and Conformal Mapping*, Am. Math. Soc., Providence, RI, 1970.

[Ber3] S. Bergman, M., Schiffer, *Kernel Functions and Elliptic Differential Equations in Mathematical Physics*, Academic Press, New York, 1953.

[Boa1] H.P. Boas, *Lu Qi-Keng's problem*, J. Korean Math. Soc., 37(2000), 253-267.

[Boa2] H.P. Boas, S. Fu, E.J. Straube, *The Bergman kernel function: explicit formula and zeroes*, Proc. Amer. Math. Soc, 127(1999), no. 3, 805-811.

[Bon-Sch] M. Bonk, O. Schramm, *Embeddings of Gromov hyperbolic spaces*, GAFA 10 (2000) 266-306.

[Bow] B.H. Bowditch, *Notes on Gromov's hyperbolicity criterion for path-metric spaces* in Group theory from a geometrical viewpoint, ed. E. Ghys, A. Haefliger, A. Verjovsky, World Scientific, Singapore, 1991, 64-167.

[Buc] S. Buckley, *Gromov hyperbolicity of invariant metrics* in Research Problems Section Complex variables, Function Spaces and Operators between them.

[Con] M. Conrad, *Nicht isotrope Abschätzungen für lineal konvexe Gebiete endlichen Typs*, Dissertation, Universität Wuppertal, 2002.

[Cos1] C. Costara, *The symmetrized bidisc as a counterexample to the converse of the Lempert's theorem*, Bull. London Math. Soc., 36(2004), 656-662.

[Cos2] C. Costara, *On the spectral Nevanlinna-Pick problem,* Studia Math., 170(2004), 23-55.

[E-K-Z] A. Edigarian, Ł. Kosiński, W. Zwonek, *The Lempert Theorem and the Tetrablock*, J. Geom. Anal., 23(2013), no. 4, 1818-831.

[Edi-Zwo] A. Edigarian, W. Zwonek, *Geometry of symmetrized polydisc*, Arch. Math.,84(2005), 364-374.

[For-Lee] J.E. Fornaess, L. Lee, *Kobayashi, Carathéodory and Sibony metric*, Complex Var. Elliptic Equ., 54(2009), 293-301.

[For-Ros] F. Forstneric, J.P. Rosay *Localization ot the Kobayashi metric and the boundary continuity of proper holomorphic mappings*, Math. Ann., 279 (1987), 239-252.

[Ful-Har] W. Fulton, J. Harris, *Representation theory. First course*, Springer Verlag, New York, 1991.

[Gam] T.W. Gamelin, *Complex analysis*, Springer, Berlin, 2000.

[Gau-Ses] H. Gaussier, H. Seshadri, *On the Gromov hyperbolicity of convex domains in* $\mathbb{C}^n$, arXiv:1312.0368.

[Gro1] M. Gromov, *Hyperbolic groups. Essays in group theory*, Math. Sci. Res. Ins. Publ., 8, Springer, New York, 1987, 75-263.

[Gro2] M. Gromov, *Carnot-Carathéodory spaces seen from within* in Sub-Riemannian geometry Eds A. Bellaïche, J.-J. Risler, Progress in Math., 144, Bierkhäuser, Basel, 1996, 75-323.

[Gun] R.C. Gunning, *Lectures on Riemann surfaces*, Princeton University Press, Princeton, New York, 1966.

[Her] G. Herbort, *Estimation of the Carathéodory distance on pseudoconvex domains of finite type, whose boundary has a Levi form of corank at most one*, Ann. Polon. Math. 109 (2013), 209-260.

[Hua] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, AMS, Providence, vol 6, 1963.

[Jak-Jar] P. Jakóbczak, M. Jarnicki, *Lectures on holomorphic functions of several complex variables*, 2001.

[Jar-Pfl1] M. Jarnicki, P. Pflug, *Automorphisms of the symmetrized bidisc*, Arch. Math. 83(2004), 264-266.

[Jar-Pfl2] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis*, Walter de Gruyter & Co., Berlin, 1993.

[Jar-Pfl3] M. Jarnicki, P. Pflug, *Extension of Holomorphic Functions*, Walter de Gruyter & Co., Berlin, 2000.

[Kos] Ł. Kosiński, *The group of automorphisms of the pentablock*, Complex Analysis & Operator Theory (to appear).

[Kra1] S.G. Krantz, *Geometric analysis of the Bergman kernel and metric*, Springer, New York, 2013.

[Kra2] S.G. Krantz, *Function theory of several complex variables*, AMS Providence, Chelsea Publishing, 2001.

[Kra3] S.G. Krantz, *The boundary behavior of the Kobayashi metric*, Rocky. Mount. J. Math., 22(1992), 227-233.

[Kra-Par] S.G. Krantz, H.R. Parks, *The geometry of domains in space*, Birkhäuser, Boston, 1999.

[Lem1] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France, 109(1981), no. 4, 427-474.

[Lem2] L. Lempert, *Holomorphic retracts and intrinsic metrics in convex domains*, Anal. Math., 8(1982), no. 4, 257-261.

[Lig] E. Ligocka, *On the Forelli-Rudin construction and the weighted Bergman projections*, Studia Math., 94(1989), 257-272.

[Lon] A. Lonjou, *Hyperboliciteé du complexe des courbes*, 2014.

[MaC] B. MacCluer, *Elemantary functional analysis*, Springer, New York, 2009.

[McN] J.D. McNeal, *Convex domains of finite type*, J. Funct. Anal., 108(1992), 361-373.

[Mah] P. Mahajan, K. Verma, *Some aspects of the Kobayashi and Carathéodory metrics on pseudoconvex domains*, J. Geom. Anal., 22(2012), 491-560.

[Mar] M. Marden, *Geometry of polynomials*, Provience, American Mathematical Society, 1966.

[McN] J.D. McNeal, *Convex domains of finite type*, J. Funct. Anal., 108(1992), 361-373.

[M-R-Z] G. Misra, S. S. Roy, G. Zhang, *Reproducing kernel for a class of weighted Bergman spaces on the symmetrized polydiscs*, Proc. Amer. Math. Soc., 2013, 141(7), 2361-2370.

[Nik1] N. Nikolov, *Estimates of invariant metrics on "convex domains"*, Ann. Mat. Pura Appl., DOI 10.1007/s10231-013-0345-7.

[Nik-Pfl] N. Nikolov, P. Pflug, *Estimates for the Bergman kernel and metric of convex domains in* $\mathbb{C}^n$, Ann. Polon. Math., 81(2003), 73-78.

[N-P-T] N. Nikolov, P. Pflug, P. J. Thomas, *Upper bound for the Lempert function of smooth domains*, Math. Z., 266(2010), 425-430.

[N-P-Z1] N. Nikolov, P. Pflug, W. Zwonek, *An example of a bounded C-convex domain which is not biholomorphic to a convex domain*, Math. Scan., 102(2008), 149-155.

[N-P-Z2] N. Nikolov, P. Pflug, W. Zwonek, *Estimates for invariant metrics on* $\mathbb{C}$*-convex domains*, Trans. Amer. Math. Soc., 363(2011), 6245-6256.

[N-T-T] N. Nikolov, P. Thomas, M. Trybuła, *Gromov (non)hyperbolicity of certain domains in* $\mathbb{C}^2$, preprint arXiv:1403.7673.

[Nik-Try1] N. Nikolov, M. Trybuła, *The Kobayashi balls of* $\mathbb{C}$*-convex domains*, Monatshefte für Mathematik (to appear), DOI:10.1007/s00605-015-0746-3.

[Nik-Try2] N. Nikolov, M. Trybuła, *Estimates of the Bergman distance on Dini-smooth bounded planar domains*, preprint arXiv:1407.6696.

[Pfl-You] P. Pflug, H. Youssif, *Complex geodesics of the minimal ball in* $\mathbb{C}^n$, Ann. Scuola Norm. Sup. Pisa, 5(3)(2004), 53-66.

[Pfl-Zwo] P. Pflug, W. Zwonek, *Logaritmic capacity and Bergman functions*, Arch. Math. 80(2003), 536-552.

[Pom1] C. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, Berlin, 1992.

[Pom2] C. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, 1975.

[Pól-Sze] G. Pólya, G. Szegö, *Problems and theorems in analysis II*, New York, Springer Verlag, 1978.

[Rud1] W. Rudin, *Proper holomorphic maps and finite reflection groups*, Indiana Univ. Math.J., 31(1982), 701-720.

[Rud2] w. Rudin, *Function theory in the unit ball*, Springer, Berlin, 1980.

[Sza] F.H. Szafraniec, *Przestrzenie Hilberta z jądrem reprodukującym*, Wyd. UJ, Kraków, 2004.

[Try1] M. Trybuła, *Proper holomorphic mappings, Bell's formula and the Lu Qi-Keng problem on the tetrablock*, Arch. Math., 2013, 101(6), 549-558.

[Try2] M.Trybuła, *Note on Costara's paper 'On the spectral Nevanlina-Pick problem'*, Cr. Acad. Bulg. Sci., 2014, 67(2), 161-166.

[Try3] M. Trybuła, *Invariant metrics on the symmetrized bidisc*, Complex Variables Theory Appl., vol. 60, Issue 4 (2015), 559-565.

[Väi] J. Väisälä, *Gromov hyperbolic spaces*, Expo. Math., 23(2005), 187-231.

[Ven] S. Venturini, *Comparison between the Kobayashi and the Carathéodory distances on strongly pseudoconvex bounded domains in* $\mathbb{C}^n$, Proc. Am. Math. Soc., 107 (1989), 725-730.

[Yin-Zha] W. Yin, L. Zhang, *Lu Qi-Keng's problem on some complex ellipsoids*, J. Math. Anal. Appl., 357(2009), 364-370.

[Zna-Zna] S.V. Znamenskii, L.N. Znamenskaya, *Spiral connectedness of the sections and projections of* $\mathbb{C}-$*convex sets*, Math. Notes, 59(1996), 253-260.

[Zwo] W. Zwonek, *Geometric properties of the tetrablock*, Arch. Math. 100(2)(2013), 159-165.